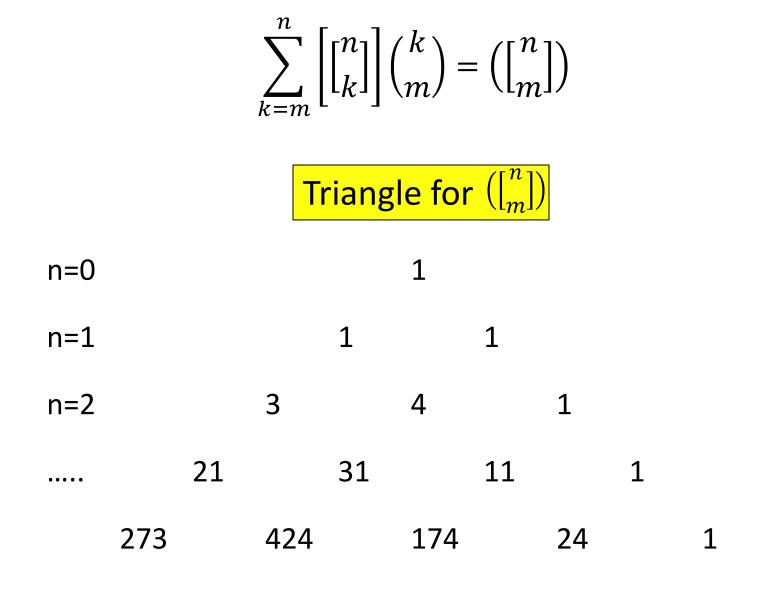
Convolutions of **Triangular Recurrences**

$$\sum_{k=m}^{n} {n \brack k} {k \choose m} = {n+1 \choose m+1}$$

This equation is known to be true for the Stirling numbers of the 1st kind. A similar equation holds for the Stirling numbers of the 2nd kind. So, how can we adapt them to fit the Legendre-Stirling numbers?

We define a function of n and m to equal the above sum. Then, we can draw up a new triangle for that function.



This new function also satisfies a recurrence

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{(n)(n-1)+1}{\binom{n-1}{m}}$$

For comparison, here is the recurrence for the Legendre-Stirling numbers of the 1st kind, which is remarkably similar.

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n)(n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix}$$

Before going any further, it's important to introduce some new notation.

$$\binom{n}{k}_{a,b} = a \binom{n-1}{k-1}_{a,b} + b \binom{n-1}{k}_{a,b}$$

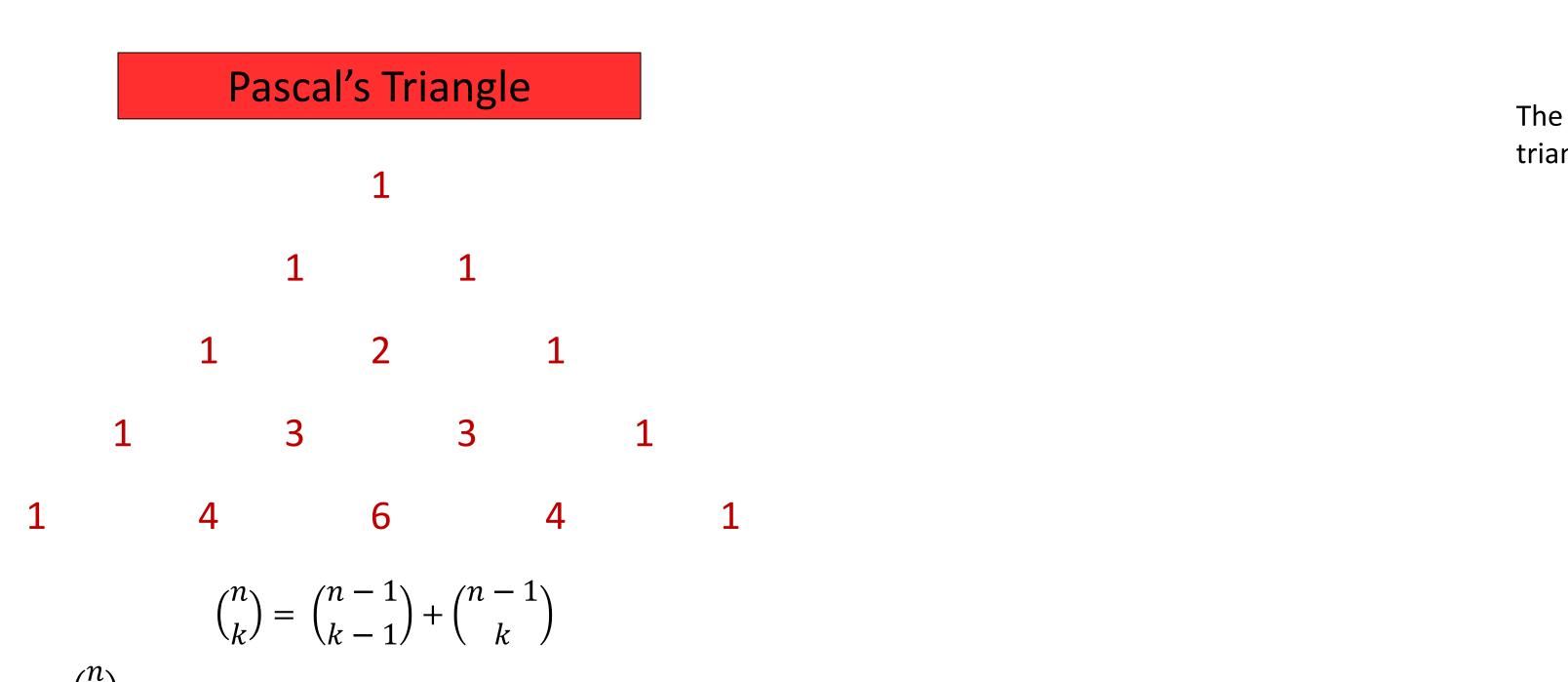
Using this notation, I can write out a generalized formula for two generic triangular recurrences.

$$\sum_{k=m}^{n} \binom{n}{k}_{a,b} \binom{k}{m}_{c,d} = \binom{n}{m}_{ac,b+ad}$$

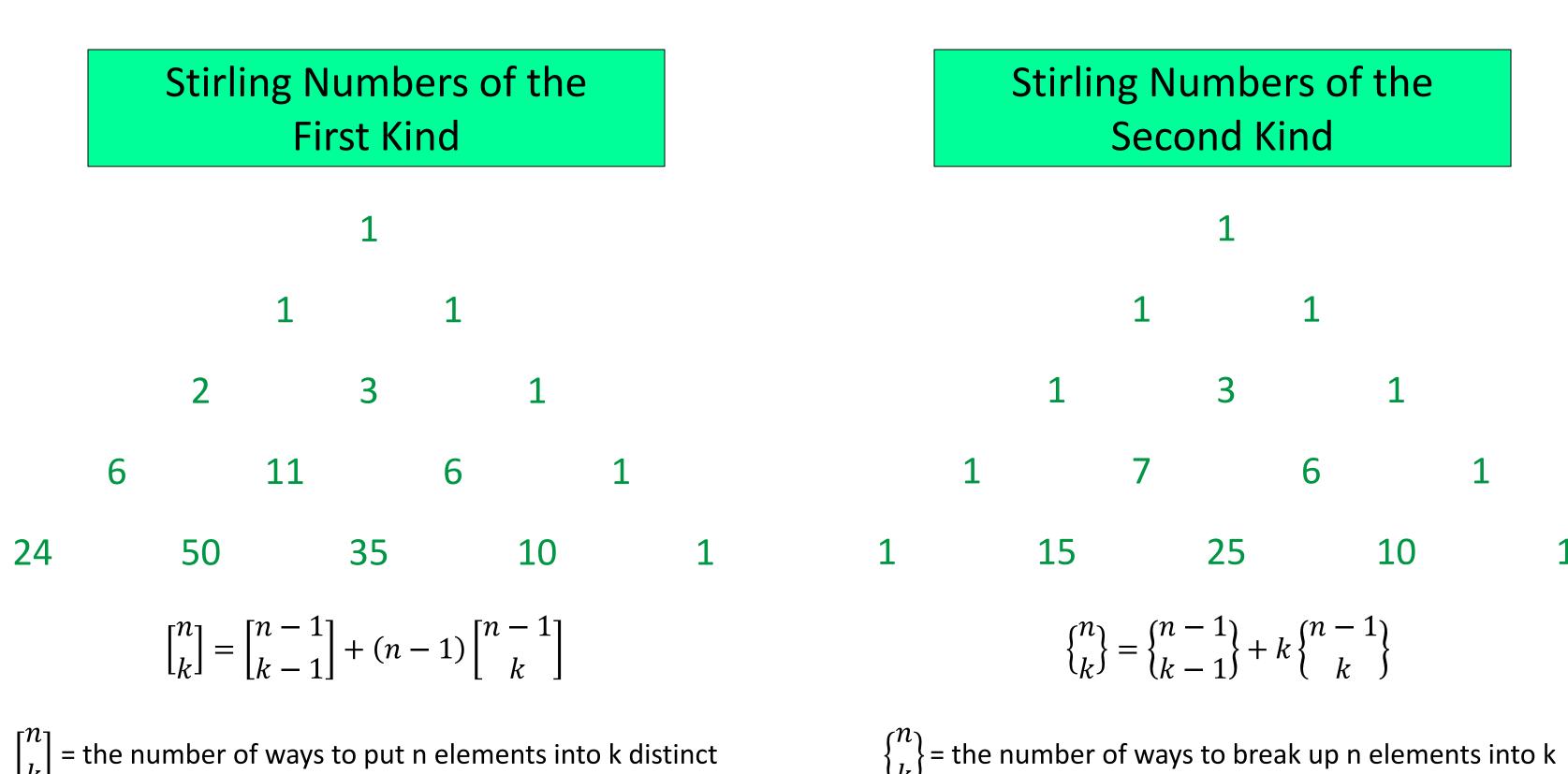
Note that a, b can be functions of n, but not of k. Also, c and d can be functions of k, not of n. For example:

$$\sum_{k=m}^{n} \binom{n}{k} \left\{ \binom{k}{m} \right\} = \binom{n}{m}_{1,k(k+1)+1}$$
$$\sum_{k=m}^{n} \left[\binom{n}{k} \right] \left\{ \binom{k}{m} \right\} = \binom{n}{m}_{1,k(k+1)+n(n-1)}$$

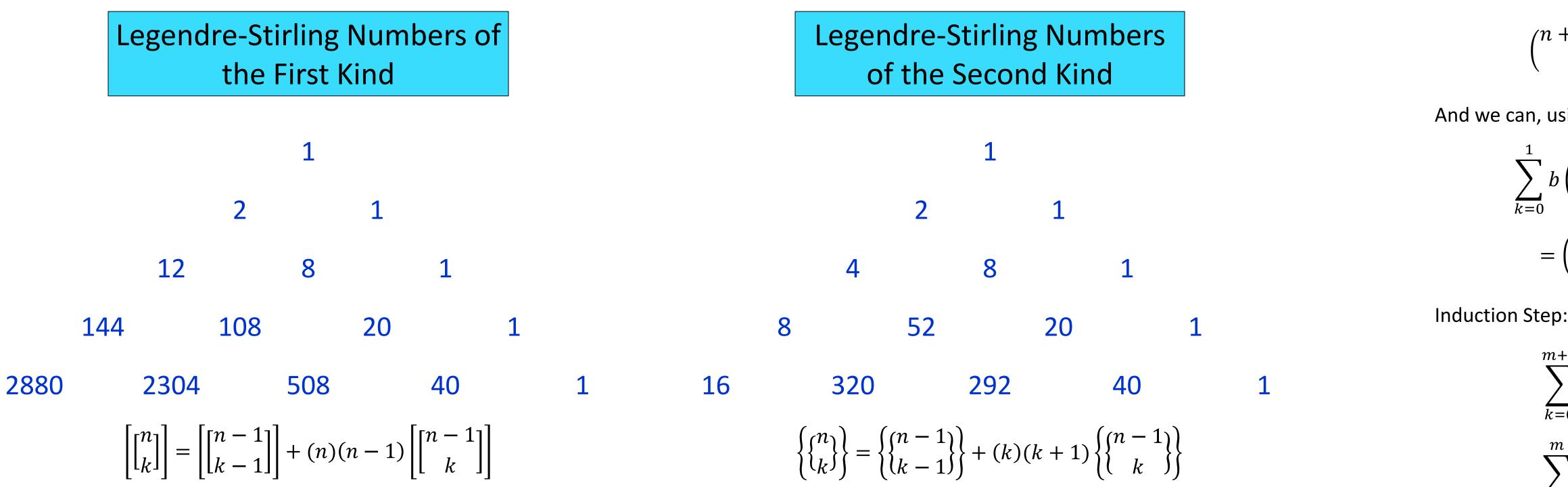
Identities for Triangular Recurrences Alex Fisher Advisor: Eric Egge



 $\langle k \rangle$ = the number of ways to choose k elements from n elements, often read as "n choose k"



cycles, often read "n cycle k"



Legendre-Stirling numbers of the 1st kind count the number of ways to choose the following:

1. You have two permutations, P_1 and P_2 .

2. P_1 is a permutation of $\{1, 2, ..., n\}$, while P_2 is a permutation of {1,2,...,n,n+1}

3. P_1 contains k cycles. P_2 contains k+1 cycles.

4. The largest element of each cycle in P_1 must also be the largest element of a cycle in P_2 .

For example, the following is one such arrangement for n=4, k=2:

$$P_1 = [1,3][2,4]$$

 $P_2 = [3][1,4][1,5]$

following:

$$\binom{n}{k} = \binom{n-1}{k-1} + k \binom{n-1}{k}$$

distinct partitions, often read as "n partition k"

$$\binom{n}{k} = \left\{ \binom{n-1}{k-1} \right\} + (k)(k+1) \left\{ \binom{n-1}{k} \right\}$$

Legendre-Stirling numbers of the 2nd kind count the number of ways to choose the

1. You have two copies of each of the elements {1,2,...,n}

2. These 2n elements are split up into k different blocks, plus one "zero-block".

3. Each non-zero block must contain both copies of its minimum element.

4. Each non-zero block cannot contain more than one copy of any element other than its minimum element.

5. The "zero-block" may not contain both copies of any of its elements.

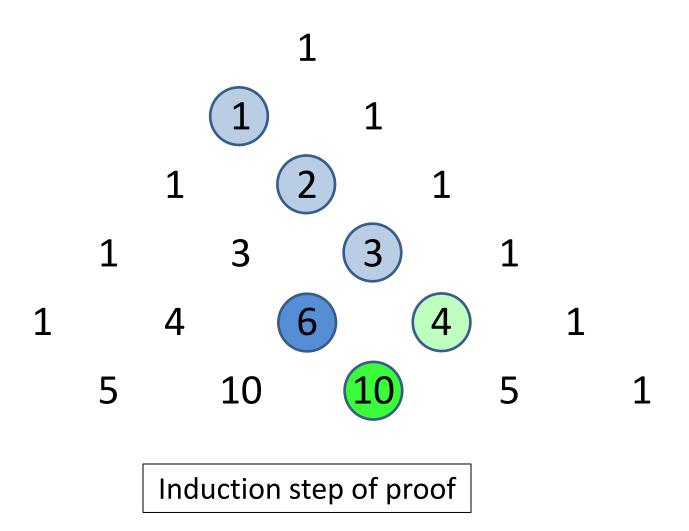
6. Only the zero-block may be empty. For example, the following is one such arrangement for n=4, k=2 :

 $\{2_2, 3_1\}, \{1_1, 1_2, 2_1, 3_2\}, \{4_1, 4_2\}$

Hockey Stick Theorem

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$$

The above formula is known as the hockey-stick theorem. The triangle below illustrates a proof of this theorem:



There is also an analog of this theorem for the Stirling numbers:

$$\sum_{k=0}^{m} (n+k) {n+k \brack k} = {n+m+1 \brack m}$$

If we rewrite the recurrences for Pascal's triangle and the Stirling numbers for n+k+1 instead of n, we see that the expression inside each summation is the right-hand side of the recurrence.

From this, we show that:

$$(n+k)(n+k+1)\left[\binom{n+k}{k}\right] = \left[\binom{n+m+1}{m}\right]$$

So the next step is to generalize the theorem to apply to all triangular recurrences of the form:

$$\binom{n+k+1}{k}_{1,b} = \binom{n+k}{k-1}_{1,b} + b\binom{n+k}{k}_{1,b}$$

And we can, using induction. The base case is n=1, m=1:

$$\sum_{k=0}^{1} b \binom{n+k}{k}_{1,b} = b \binom{n}{0}_{1,b} + b \binom{n+1}{1}_{1,b} = \left(\binom{n+1}{0}_{1,b} + b \binom{n+1}{1} \right) = \left(\binom{n+2}{1}_{1,b} \right)_{1,b}$$

$$\sum_{k=0}^{m+1} b \binom{n+k}{k}_{1,b} =$$

$$\sum_{k=0}^{m} b \binom{n+k}{k}_{1,b} + b \binom{n+m+1}{m+1}_{1,b} =$$

$$\binom{n+m+1}{m}_{1,b} + b \binom{n+m+1}{m+1}_{1,b} = \binom{n+m+2}{m+1}_{1,b}$$

When generalized even further, we get:

$$\sum_{k=0}^{m} a^{m-k} b \binom{n+k}{k}_{a,b} = \binom{n+m+1}{m}_{a,b}$$

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