Symmetric Permutations with No Long Decreasing Subsequences

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Permutations and Pattern Avoidance

Definition

π, σ are permutations.

π avoids σ whenever π has no subsequence with same length and relative order as σ.

Example

6152347 avoids 231 but not 213.

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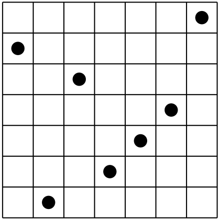

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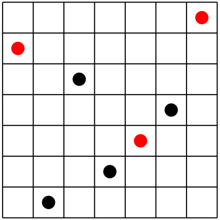

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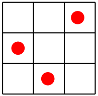

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How many permutations in S_n

- *avoid a given set R of patterns and*
- *are invariant under* 180° *rotation?*

Denote this set by $S_n^{rc}(R)$.

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Question

What is $|S_{2n}^{rc}(4321)|$?

$$\pi \mapsto (P(\pi), Q(\pi))$$

Example

$$\pi = 52413$$

1	3
2	4
5	

$P(\pi)$

1	3
2	5
4	

$Q(\pi)$

The RSK Correspondence

$$\pi \mapsto (P(\pi), Q(\pi))$$

Example

$$\pi = 52413$$

1	3
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$P(\pi)$

1	3
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$Q(\pi)$

Theorem (Schensted)

The length of the longest decreasing subsequence in π is the number of rows in $P(\pi)$.

Evacuation

$$ev : standard\ \ tableaux \longrightarrow standard\ \ tableaux$$

$$ev \circ ev = identity$$

Example

1	3
2	5
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P

$ev(P)$

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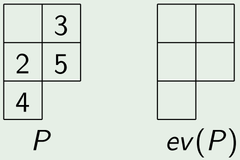

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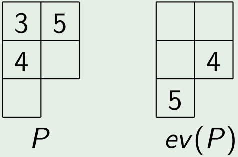

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Example

P:

1	3
2	5
4	

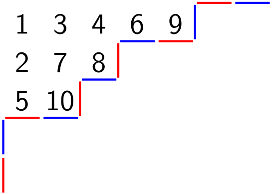
$ev(P)$:

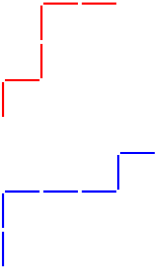
1	2
3	4
5	

Theorem

π is invariant under 180° rotation if and only if $ev(P(\pi)) = P(\pi)$ and $ev(Q(\pi)) = Q(\pi)$.

Factoring Self-Evacuating Tableaux





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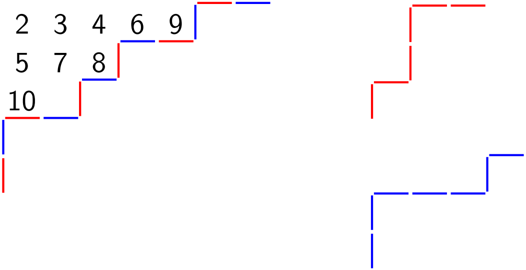

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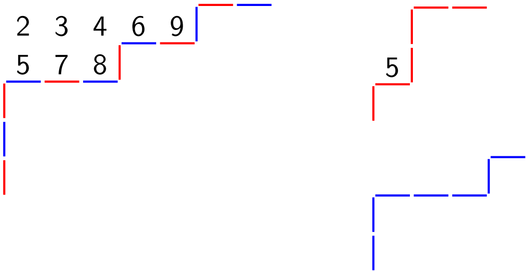

At each step we

- evacuate,
- remove the largest element,
- place next number in appropriate square.

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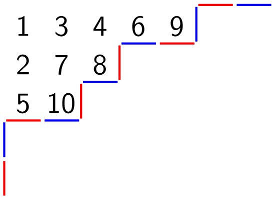

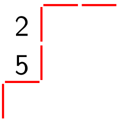

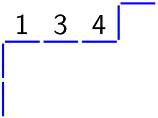

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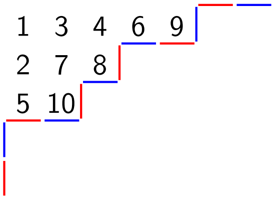

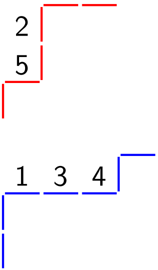

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Fact

If P has $2n$ entries and at most 3 rows then

- *the entries of P_r and P_b partition $[n]$;*
- *P_r has at most 2 rows;*
- *P_b has at most 1 row.*

Putting it all together

$$\pi \mapsto (P(\pi), Q(\pi)) \mapsto (P_r(\pi), P_b(\pi), Q_r(\pi), Q_b(\pi)) \mapsto E_1, E_2, \pi_r, \pi_b$$

- $\pi \in S_{2n}^{rc}(4321)$

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- $P(\pi)$, $Q(\pi)$ have $2n$ entries and at most 3 rows.
- P_r, P_b have entries $[n]$ and Q_r, Q_b have entries $[n]$.
- P_r, Q_r each has at most 2 rows and P_b, Q_b each has at most 1 row.

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- $E_1, E_2 \subseteq [n]$ and $|E_1| = |E_2|$.
- $\pi_r \in S_{|E_1|}(321)$ and $\pi_b \in S_{|E_1|}(21)$.

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Theorem (E,2010)

$$|S_{2n+1}^{rc}(4321)| = |S_{2n}^{rc}(4321)| = \sum_{j=0}^{n} \binom{n}{j}^2 C_j,$$

Results for $S_n^{rc}(k\ldots 321)$

Theorem (E,2010)

For all $k \geq 2$ and all $n \geq 0$ we have

$$|S_{2n}^{rc}(k\ldots 21)| = \sum_{j=0}^{n} \binom{n}{j}^2 \left|S_j\left(\left\lceil\frac{k+1}{2}\right\rceil\ldots 21\right)\right| \left|S_{n-j}\left(\left\lfloor\frac{k+1}{2}\right\rfloor\ldots 21\right)\right|.$$

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Corollary

$$|S_{2n}^{rc}(54321)| = \sum_{j=0}^{n} \binom{n}{j}^2 C_j C_{n-j},$$

The End

Thank You!