Snow Leopard Permutations, Even Knots, Odd Knots, Janus Knots, and Restricted Catalan Paths

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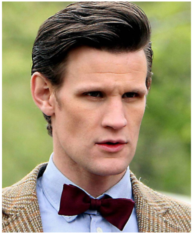
My Students


Ben


Greg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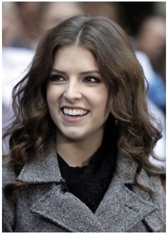
Kailee

A Problem in Analysis

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$$f(g(t)) = g(f(t)).$$

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f and g must have a common fixed point.

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Observation (Baxter, 1964)

f and g permute the fixed points of $g \circ f = f \circ g$.

Complete Baxter Permutations

Definition

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Example

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Baxter Permutations and anti-Baxter Permutations

Permutation in the odd entries:

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- Commonly called a (reduced) Baxter permutation
- Is characterized by avoiding the generalized patterns $3-14-2$ and $2-41-3$

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Permutation in the even entries:

- May not determine a unique complete Baxter permutation
- Has no common name, though sometimes called an *anti-Baxter* permutation
- Is characterized by avoiding the generalized patterns $3-41-2$ and $2-14-3$

Compatibility

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If there exists a complete Baxter permutation π such that π_1 and π_2 are the permutations induced on the odd and even entries of π, respectively, we say that π_1 and π_2 are *compatible*.

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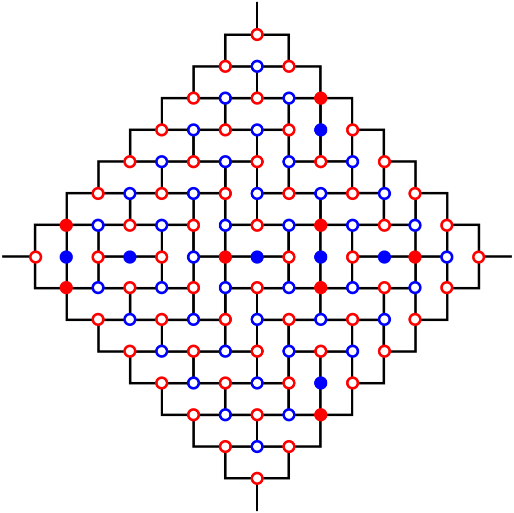
Products of Fibonacci Numbers

Theorem

The number of Baxter permutations compatible with a given anti-Baxter permutation is a product of Fibonacci numbers.

Aztec Diamond



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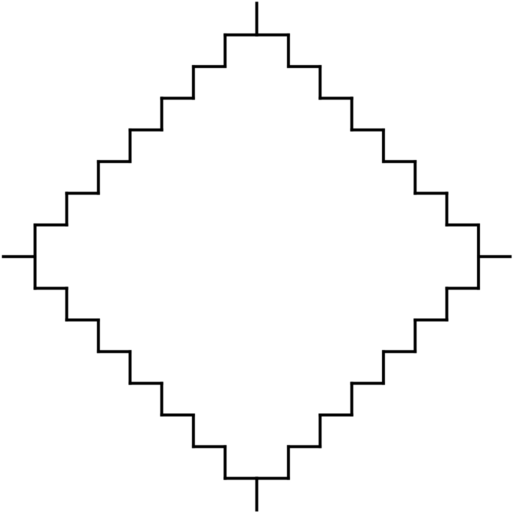

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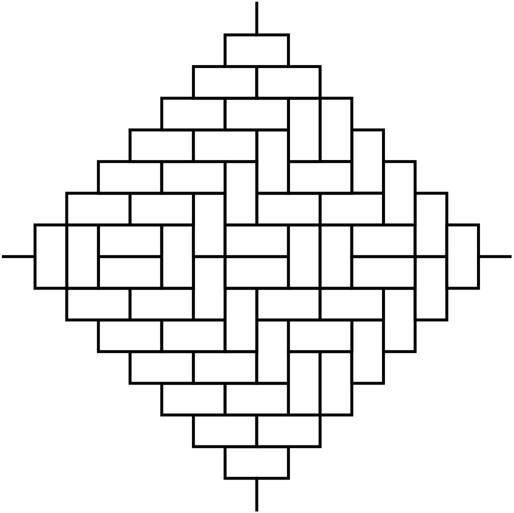

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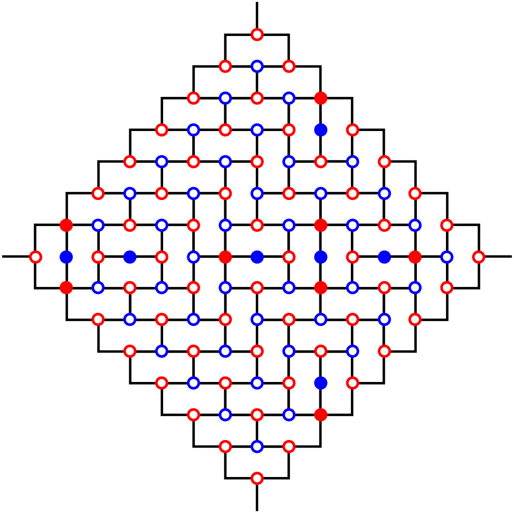

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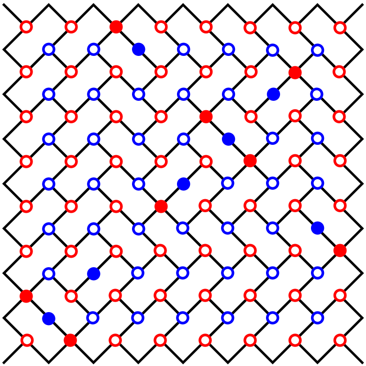

DABPs

Doubly Alternating Baxter Permutations

- ascents and descents alternate in π, beginning with an ascent
- ascents and descents alternate in π^{-1}, beginning with an ascent
- Baxter

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Theorem (Guibert & Linusson, 2000)

The number of DABPs of length $2n$ is C_n, the n^{th} Catalan number.

Snow Leopard Permutations

Definition

We call the permutations of length n which are compatible with the DABPs of length $n+1$ the *snow leopard permutations* (SLPs).

Examples

1
123, 321
12345, 14325, 34521, 54123, 54321

Properties

- anti-Baxter
- identity and reverse identity are always snow leopard
- odd entries in odd positions, even entries in even positions

Decomposition of SLPs

Theorem (Caffrey, Egge, Michel, Rubin, Ver Steegh)

A permutation π of length $2n$ is an SLP if and only if there exists an SLP σ of length $2n-1$ such that $\pi = 1 \oplus \sigma^c$.

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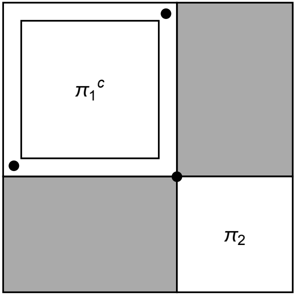

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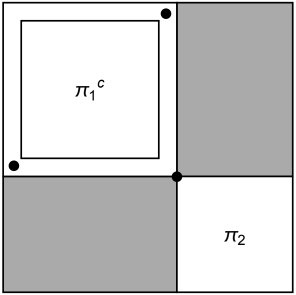

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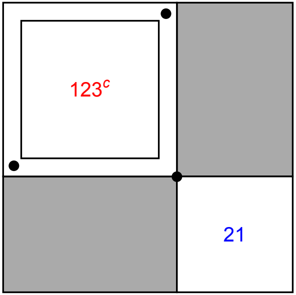

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$(1 \oplus 123^c \oplus 1) \ominus 1 \ominus 321$

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Theorem (Caffrey, Egge, Michel, Rubin, Ver Steegh)

$SL_n :=$ the set of snow leopard permutations of length $2n-1$

- $|SL_1| = 1, \ |SL_2| = 2$
- $|SL_{n+1}| = \sum_{j=0}^{n} |SL_j||SL_{n-j}|$
- $|SL_n| = C_n$

Bijection with Catalan paths

3 6 5 4 7 2 1

Bijection with Catalan paths

8 3 6 5 4 7 2 1 0

Bijection with Catalan paths

8		3		6		5		4		7		2		1		0
	d		a		d		d		a		d		d		d	

Bijection with Catalan paths

8		3		6		5		4		7		2		1		0
	d		a		d		d		a		d		d		d	
	N		E		N		N		E		N		N		N	

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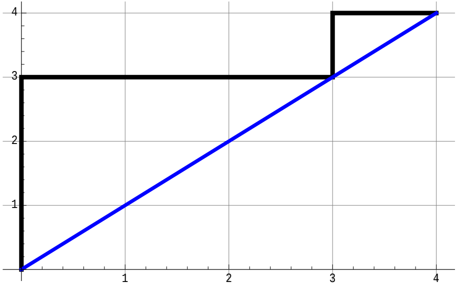
	d		a		d		d		a		d		d		d	
	N		N		N		E		E		E		N		E	

Bijection with Catalan paths

8 3 6 5 4 7 2 1 0

d a d d a d d d

N N N E E E N E



Odd and Even Knots

Definition

We call the permutation induced on the even entries of an SLP π an *even knot* (even(π)) and the permutation induced on the odd entries an *odd knot* (odd(π)).

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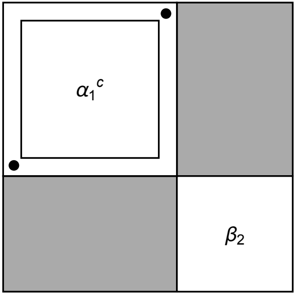
Examples

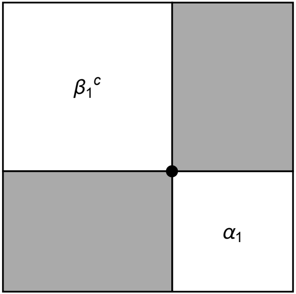
Odd knots: $\emptyset$, 1, 12, 21, 123, 231, 312, 321

Even knots: $\emptyset$, 1, 12, 21, 123, 132, 213, 231, 312, 321

Decomposition of Even and Odd Knots



Odd knot β decomposition



Even knot α decomposition

What are the odd and even knots counted by?

n	0	1	2	3	4	5	6
$\lvert EK_n \rvert$	1	1	2	6	17	46	128
$\lvert OK_n \rvert$	1	1	2	4	9	23	63

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Theorem (Egge, Rubin)

The even knots of length n are in bijection with the set of Catalan paths of length $n+1$ which have no ascent of length exactly 2. (Essentially no ENNE.)

Entangled Knots

Definition

We say an even knot α and an odd knot β are *entangled* whenever there exists an SLP π such that $\text{even}(\pi) = \alpha$ and $\text{odd}(\pi) = \beta$.

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The even knots of length $n-1$ entangled with the identity permutation of length n are the 3412-avoiding involutions of length $n-1$.

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Theorem (Egge, Rubin)

The odd knots of length $n+1$ entangled with the reverse identity permutation of length n are the complements of the 3412-avoiding involutions of length $n+1$.

Motzkin Numbers

M_n is the number of lattice paths from $(0, 0)$ to $(n, 0)$ using only up $(1, 1)$, level $(1, 0)$, and down $(1, -1)$ steps.

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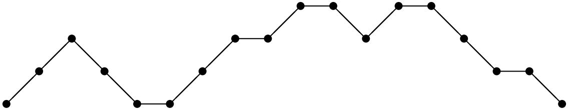

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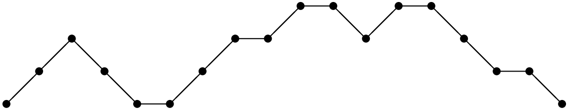

n	0	1	2	3	4	5	6	7	8	9
M_n	1	1	2	4	9	21	51	127	323	835

Entangled Knots

Corollary (Egge, Rubin)

The number of even knots of length $n-1$ entangled with the identity permutation of length n is M_{n-1}, where M_n is the n^{th} Motzkin number.

Corollary (Egge, Rubin)

The number of odd knots of length $n+1$ entangled with the reverse identity permutation of length n is M_{n+1}.

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Conjecture

For each even (resp. odd) knot, the number of entangled odd (resp. even) knots is a product of Motzkin numbers.

Janus Knots

Odd Knots	Even Knots
1	1
12	12
21	21
123	123
231	132
312	213
321	231
1234	312
1324	321
2341	1234
3412	1243
3421	1324
4123	1432
4231	2134
4312	2143
4321	2341
12345	2431
12435	3214

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Janus Knots

Definition

A *janus knot* is a permutation which is both an even knot and an odd knot.

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Janus Knots

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n	1	2	3	4	5	6	7	8	9
$\lvert J_n \rvert$	1	2	4	8	17	37	82	185	423

Even Knots

1
12
21
123
132
213
231
312
321
1234
1243
1324
1432
2134
2143
2341
2431
3214

Janus Knots and Motzkin Paths

A004148 Generalized Catalan numbers: a(n+1) = a(n)+ Sum(k=1..n-1, a(k)*a(n-1-k)). 169
(Formerly M1141)

1, 1, 1, 2, 4, 8, 17, 37, 82, 185, 423, 978, 2283, 5373, 12735, 30372, 72832, 175502, 424748, 1032004, 2516347, 6155441, 15101701, 37150472, 91618049, 226460893, 560954047, 1392251012, 3461824644, 8622571758, 21511212261, 53745962199, 134474581374 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,4

COMMENTS

Arises in enumerating secondary structures of RNA molecules. The 17 structures with 6 nucleotides are shown in the illustration (after Waterman, 1978).
Hankel transform is period 8 sequence [1,1,1,0,-1,-1,-1,0,...].
Enumerates peak-less Motzkin paths of length n. Example: a(5)=8 because we have HHHHH, HHUHD, HUHDH, HUHHD, UHDHH, UHHDH, UHHHD, UUHDD, where U=(1,1), D=(1,-1) and H=(1,0). - Emeric Deutsch, Nov 19 2003
Number of Dyck paths of semilength n-1 with no UUU's and no DDD's, where U=(1,1) and D=(1,-1) (n>0) - Emeric Deutsch, Nov 19 2003
For n>=1, a(n) = number of dissections of an (n+2)-gon with strictly disjoint diagonals and no diagonal incident with the base. (One side of the (n+2)-gon is designated the base.) - David Callan, Mar 23 2004
For n>=2, a(n-2)= number of UU-free Motzkin n-paths = number of DU-free Motzkin n-paths. - David Callan, Jul 15 2004
a(n)=number of UU-free Motzkin n-paths containing no low peaks (A low peak is a UD pair at ground level, i.e. whose removal would create a pair of Motzkin paths). For n>=1, a(n)=number of UU-free Motzkin (n-1)-paths = number of DU-free Motzkin (n-1)-paths. a(n) is asymptotically ~ c n^(-3/2) (1 + phi)^n with c = 1.1043... and phi=(1+sqrt(5))/2. - David Callan, Jul 15 2004. In closed form, c = sqrt(30+14*sqrt(5))/(4*sqrt(Pi)) = 1.104365547309692849... - Vaclav Kotesovec, Sep 11 2013
a(n) = number of Dyck (n+1)-paths with all pyramid sizes >= 2. A pyramid is a maximal subpath of the form k upsteps immediately followed by k downsteps and its size is k. - David Callan, Oct 24 2004
a(n)=number of Dyck paths of semilength n+1 with no small pyramids (n>=1). A pyramid is a maximal sequence of the form k Us followed by k Ds with k>=1. A small pyramid is one with k=1. For example, a[4]=4 counts the following Dyck 5-paths (pyramids denoted by lowercase letters and separated by a vertical bar): uuuuuddddd, Uuudd|uuddD, uudd|uuuddd, uuuddd|uudd. - David Callan, Oct 25 2004
From Emeric Deutsch, Jan 08 2006: (Start)
"a(n)=number of Motzkin paths of length n-1 with no peaks at level >=1.

Theorem (Egge, Rubin)

There is a natural bijection between the set of janus knots of length n and the set of peakless Motzkin paths of length $n+1$.

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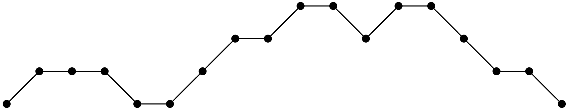

The Last Slide

Thank you!