Enumerating rc-Invariant Permutations with No Long Decreasing Subsequences*

Eric S. Egge
Department of Mathematics
Carleton College
Northfield, MN 55057 USA

eegge@carleton.edu

Abstract

We use the Robinson-Schensted-Knuth correspondence and Schützenberger's evacuation of standard tableaux to enumerate permutations and involutions which are invariant under the reverse-complement map and which have no decreasing subsequences of length k. These enumerations are in terms of numbers of permutations with no decreasing subsequences of length approximately $\frac{k}{2}$; we use known results concerning these quantities to give explicit formulas when $k \leq 6$.

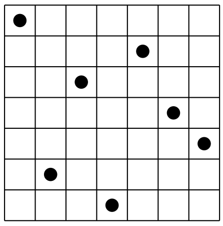
Keywords: Domino tableaux, pattern-avoiding permutation, restricted permutation, reverse-complement map, Robinson-Schensted-Knuth correspondence, tableaux.

1 Introduction

Let S_n denote the set of permutations of $\{1, 2, \ldots, n\}$, written in one-line notation, and suppose π and σ are permutations. We say π *contains* σ whenever π has a subsequence with the same length and relative order as σ; otherwise we say π *avoids* σ. For example, 4257316 avoids 1432, but it has 436 as a subsequence so it contains 213. In this context we sometimes call π a *pattern-avoiding permutation* or a *restricted permutation*, and we sometimes call σ a *pattern* or a *forbidden pattern*.

It is well-known that the symmetry group of a square, which we denote by D_4, acts on permutations in a way that is compatible with pattern avoidance. To describe this action on a given permutation $\pi \in S_n$, we first draw the *diagram of* π, as follows. Begin with an $n \times n$ square, subdivided into n rows of equal height and n columns of equal width. Number the rows (resp. columns) $1, 2, \ldots, n$ from left to right (resp. bottom to top), and place dots in the 1×1 squares with (row,column) coordinates $(1, \pi(1)), (2, \pi(2)), \ldots, (n, \pi(n))$, where $\pi(i)$ is the ith entry of π. For example, the diagram of 7251643 is pictured below.

*2000 Mathematics Subject Classification: Primary 05A05, 05A15; Secondary 30B70, 42C05



The diagram of 7251643.

The group D_4 acts naturally on these diagrams, and therefore on permutations; for any $g \in D_4$ we write π^g to denote the image of π under g. We observe that $\pi \in S_n$ contains $\sigma \in S_k$ if and only if there exist k rows and k columns in the diagram of π whose squares of intersection form the diagram of σ. It follows that if $g \in D_4$ then π avoids σ if and only if π^g avoids σ^g. In this context it is traditional to single out the following three elements of D_4.

- The reverse map r reverses the order of the entries of π. On diagrams r is the reflection over a vertical line.
- The complement map c replaces each entry $\pi(j)$ of $\pi \in S_n$ with $n + 1 - \pi(j)$. On diagrams c is the reflection over a horizontal line.
- The inverse map i takes a permutation π to its group-theoretic inverse. On diagrams i is the reflection over the diagonal from the lower left corner to the upper right corner.

We note that r, c, and i together generate D_4, each of them has order two, $rc = cr$ is in the center of D_4, $ri = ic$, and $ci = ir$.

For any subgroup $H \subseteq D_4$ and any set R of permutations, let $S_n^H(R)$ denote the set of permutations in S_n which are invariant under every element of H and which avoid every pattern in R. The set $S_n^H(R)$ has been studied for a variety of R when $H = \{e\}$ or $H = \{e, i\}$; among the first results along these lines are those of Simion and Schmidt [15]. More recently, Egge [4] has enumerated $S_n^H(R)$ for other subgroups $H \subseteq D_4$ when R consists of relatively small patterns. Often these results include refined enumerations, whose answers are given naturally in terms of binomial coefficients. Building on Egge's work, Lonoff and Ostroff [10] have enumerated $S_n^H(R)$ for $H = \{e, rc\}$ and almost all R consisting of one pattern of length 3 and one pattern of length 4. Their answers include the Fibonacci numbers, powers of 2, perfect squares, and triangular numbers, and their work has led to a Fibonacci identity which appears to be new.

In this paper we study $S_n^H(k \ldots 21)$ when $H = \{e, rc\}$ or $H = \{e, i, rc, rci\}$; throughout we abbreviate $S_n(R) = S_n^{\{e\}}(R)$, $S_n^{rc}(R) = S_n^{\{e,rc\}}(R)$, $I_n(R) = S_n^{\{e,i\}}(R)$, and $I_n^{rc}(R) = S_n^{\{e,rc,i,rci\}}(R)$. Using the Robinson-Schensted-Knuth correspondence, Schützenberger's evacuation map, and a known factorization of domino tableaux, we show that for all $k \geq 2$ and all $n \geq 0$,

$$|S_{2n}^{rc}(k \ldots 21)| = \sum_{j=0}^{n} \binom{n}{j}^2 \left| S_j \left(\left\lceil \frac{k+1}{2} \right\rceil \ldots 21 \right) \right| \left| S_{n-j} \left(\left\lfloor \frac{k+1}{2} \right\rfloor \ldots 21 \right) \right|. \tag{1}$$

Similarly, we show that when $k \geq 2$ is even and $n \geq 0$ we have

$$|S^{rc}_{2n+1}(k \ldots 21)| = \sum_{j=0}^{n} \binom{n}{j}^2 \left| S_j \left(\frac{k}{2} \ldots 21 \right) \right| \left| S_{n-j} \left(\frac{k+2}{2} \ldots 21 \right) \right|, \tag{2}$$

and when $k > 2$ is odd and $n \geq 0$ we have

$$|S^{rc}_{2n+1}(k \ldots 21)| = \sum_{j=0}^{n} \binom{n}{j}^2 \left| S_j \left(\frac{k+3}{2} \ldots 21 \right) \right| \left| S_{n-j} \left(\frac{k-1}{2} \ldots 21 \right) \right|. \tag{3}$$

We then combine these results with known enumerations of $S_n(k \ldots 21)$ to give several new enumerations of $S^{rc}_n(k \ldots 21)$, including

$$|S^{rc}_{2n+1}(4321)| = |S^{rc}_{2n}(4321)| = \sum_{j=0}^{n} \binom{n}{j}^2 C_j \qquad (n \geq 0)$$

and

$$|S^{rc}_{2n}(54321)| = \sum_{j=0}^{n} \binom{n}{j}^2 C_j C_{n-j} \qquad (n \geq 0).$$

Here, and throughout the paper, we write C_n to denote the nth Catalan number, which is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Turning our attention to $I^{rc}_n(k \ldots 21)$, we use the same techniques as in the proofs of (1), (2), and (3) to show that for all $k \geq 2$ and all $n \geq 0$,

$$|I^{rc}_{2n}(k \ldots 21)| = \sum_{j=0}^{n} \binom{n}{j} \left| I_j \left(\left\lceil \frac{k+1}{2} \right\rceil \ldots 21 \right) \right| \left| I_{n-j} \left(\left\lfloor \frac{k+1}{2} \right\rfloor \ldots 21 \right) \right|.$$

Similarly, we show that when $k \geq 2$ is even and $n \geq 0$ we have

$$|I^{rc}_{2n+1}(k \ldots 21)| = \sum_{j=0}^{n} \binom{n}{j} \left| I_j \left(\frac{k}{2} \ldots 21 \right) \right| \left| I_{n-j} \left(\frac{k+2}{2} \ldots 21 \right) \right|,$$

and when $k > 2$ is odd and $n \geq 0$ we have

$$|I^{rc}_{2n+1}(k \ldots 21)| = \sum_{j=0}^{n} \binom{n}{j} \left| I_j \left(\frac{k+3}{2} \ldots 21 \right) \right| \left| I_{n-j} \left(\frac{k-1}{2} \ldots 21 \right) \right|.$$

We then combine these results with known enumerations of $I_n(k \ldots 21)$ to give several new enumerations of $I^{rc}_n(k \ldots 21)$, including

$$|I^{rc}_{2n+1}(4321)| = |I^{rc}_{2n}(4321)| = \sum_{j=0}^{n} \binom{n}{j} \binom{j}{\lfloor \frac{j}{2} \rfloor} \qquad (n \geq 0)$$

and

$$|I^{rc}_{2n+1}(54321)| = C_{n+1} \qquad (n \geq 0).$$

2 Standard Tableaux and the RSK Correspondence

Suppose $a_1 < a_2 < \cdots < a_n$ are positive integers and $\lambda \vdash n$ (that is, λ is a partition of n). We call a filling of the Ferrers diagram of λ (oriented in the English style) with $a_1, \ldots, a_n$ a *tableau* whenever its entries increase from left to right across rows and increase from top to bottom down columns, and we call a tableau *standard* whenever $a_i = i$ for $1 \le i \le n$. In this context we call λ the *shape* of the tableau. In this section we briefly recall the Robinson-Schensted-Knuth (RSK) correspondence between permutations $\pi \in S_n$ and ordered pairs (P, Q) of standard tableaux of the same shape $\lambda \vdash n$.

Given $\pi \in S_n$, the RSK correspondence builds ordered pairs (P, Q) of standard tableaux of the same shape one entry at a time, by inserting the entries of π (read from left to right) into P and using Q to record where boxes are added to P at each step. That is, the RSK correspondence builds sequences $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ of tableaux, in which P_i (resp. Q_i) is obtained from P_{i-1} (resp. Q_{i-1}) by adding an appropriate box. To begin this process, P_1 is the tableau with a single box with entry $\pi(1)$ and Q_1 is the tableau with a single box with entry 1. To obtain P_i from P_{i-1}, we compare $\pi(i)$ with the entries in the first row of P_{i-1}. If $\pi(i)$ is larger than all such entries then we add a box to the end of the row with entry $\pi(i)$. If $\pi(i)$ is smaller than one of these entries then we replace the smallest such entry a with $\pi(i)$. Now we apply the same process to a and the next row, repeating these steps until we have added a box to the tableau. To obtain Q_i from Q_{i-1} we add a box with entry i in the same position as the box we added to P_{i-1} to obtain P_i.

Definition 2.1. *If $\pi \in S_n$ then we write $P(\pi)$ (resp. $Q(\pi)$) to denote the standard tableau P_n (resp. Q_n) obtained by the RSK correspondence described above.*

Example 2.2. *If $\pi = 35412$ then the sequences P_1, P_2, P_3, P_4, P_5 and Q_1, Q_2, Q_3, Q_4, Q_5 are as in the following table.*

i	*1*	*2*	*3*	*4*	*5*
P_i	$\begin{array}{c} 3 \end{array}$	$\begin{array}{cc} 3 & 5 \end{array}$	$\begin{array}{cc} 3 & 4 \\ 5 & \end{array}$	$\begin{array}{cc} 1 & 4 \\ 3 & \\ 5 & \end{array}$	$\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & \end{array}$
Q_i	$\begin{array}{c} 1 \end{array}$	$\begin{array}{cc} 1 & 2 \end{array}$	$\begin{array}{cc} 1 & 2 \\ 3 & \end{array}$	$\begin{array}{cc} 1 & 2 \\ 3 & \\ 4 & \end{array}$	$\begin{array}{cc} 1 & 2 \\ 3 & 5 \\ 4 & \end{array}$

The following result is well-known, so we state it without proof.

Theorem 2.3. *([12, Thm. 3.1.1]) For all $n \ge 1$, the map $\pi \mapsto (P(\pi), Q(\pi))$ given by the RSK correspondence is a bijection between S_n and the set of ordered pairs (P, Q) of standard tableaux of the same shape with n boxes.*

Many authors have written about the RSK correspondence and its properties, but its name comes from work of Robinson [11], Schensted [13], and Knuth [8]. Our interest in the RSK correspondence is rooted in the following result of Schensted [13], which has since been generalized by Greene [7].

Theorem 2.4. *([12, Thm. 3.3.2]) Suppose $\pi \in S_n$. Then the number of rows of $P(\pi)$ (and thus of $Q(\pi)$) is equal to the length of the longest decreasing subsequence of π.*

3 Evacuation and rc

In view of Theorems 2.3 and 2.4, to enumerate $S_n^{rc}(k \dots 21)$ we need to enumerate ordered pairs (P, Q) of standard tableaux of the same shape which have at most $k-1$ rows, and whose corresponding permutations are invariant under rc. In this section we describe a combinatorial operation on tableaux called evacuation, which we use to characterize $P(\pi)$ and $Q(\pi)$ when π is invariant under rc. Evacuation uses Schützenberger's jeu de taquin [14]; our account is inspired by the exposition of van Leeuwen [9]. We begin with some operations on standard tableaux.

Suppose P is a standard tableau of shape $\lambda \vdash n$. We may produce a slightly smaller standard tableau from P as follows. First, remove the 1, so that the box in the upper left corner of P is empty. Next, compare the number to the right of the empty box with the number immediately below the empty box, and move the smaller of these two numbers into the empty box. If only one of these two positions contains a number, then move that number into the empty box. Repeat this process on each new empty box, as long as the empty box has an entry to its right or immediately below it. When the empty box has no entry to its right and no entry below, remove that box from P, and subtract 1 from every number in the resulting tableau.

Definition 3.1. *For any standard tableau P with n boxes, we write $\mathrm{e}(P)$ to denote the standard tableau with $n-1$ boxes obtained by the process described in the paragraph above.*

Example 3.2. *If P is the standard tableau below on the left then $\mathrm{e}(P)$ is the standard tableau below on the right.*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 5 & 8 \\ \hline 4 & 7 \\ \cline{1-2}\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 & 7 \\ \hline 3 \\ \cline{1-1}\end{array}$$

We can invert the action of e on a standard tableau in the following way. Suppose P is a standard tableau of shape $\lambda \vdash n$, and let c denote a box not in P for which $P \cup \{c\}$ is a legal Ferrers diagram. Then c has a box above it or a box to its left; choose the larger of the numbers in these boxes and move that number into the empty box. If only one of these two positions contains a number, then move that number into the empty box. Repeat this process on the resulting diagram until the empty box is in the upper left corner. Add one to each entry of the resulting tableau, and put a 1 in the empty box.

Definition 3.3. *For any standard tableau P with n boxes and any box c not in P for which $P \cup \{c\}$ is a legal Ferrers diagram, we write $\mathrm{i}_c(P)$ to denote the standard tableau with $n+1$ boxes obtained from P by the process described in the paragraph above.*

Example 3.4. *If P and c are the standard tableau and box given below on the left, then $\mathrm{i}_c(P)$ is the standard tableau given below on the right.*

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 & c \\ \cline{1-2}\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2}\end{array}$$

It is well-known (see [12, p. 114]) that e and i_c are inverses of one another. In particular, we have the following.

Proposition 3.5. *If P is a standard tableau with n boxes then the following hold.*

(i) *If c is the box removed from P in creating* $\mathrm{e}(P)$ *then* $\mathrm{i}_c(\mathrm{e}(P)) = P$.

(ii) *If c is a box not in P for which $P \cup \{c\}$ is a legal Ferrers diagram then* $\mathrm{e}(\mathrm{i}_c(P)) = P$.

Repeatedly applying e to a given standard tableau and recording the results allows us to produce a new standard tableau of the same shape as the original one. In particular, suppose P is a standard tableau of shape $\lambda \vdash n$, and let Q be a copy of the Ferrers diagram of λ in which every box is empty. We may turn Q into a standard tableau of shape λ as follows. First apply e to P to remove a box from P, and then place n in the corresponding box of Q. Repeat this process on $\mathrm{e}(P)$, placing $n-1$ in the box of Q corresponding to the next box of P to be removed. Continue in this fashion until P has been emptied.

Definition 3.6. *For any standard tableau of shape $\lambda \vdash n$, we write* $\mathrm{ev}(P)$ *to denote the* evacuation of P, *which is the standard tableau of shape λ obtained by the process described in the paragraph above. We say P is* self-evacuating *whenever* $\mathrm{ev}(P) = P$.

Example 3.7. *If P is the standard tableau below on the left, then* $\mathrm{ev}(P)$ *is the standard tableau below on the right.*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2}\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array}$$

We can now characterize those ordered pairs $(P(\pi), Q(\pi))$ of standard tableaux for which π is invariant under rc.

Theorem 3.8. *If $\pi \in S_n$ then $P(\pi^{rc}) = \mathrm{ev}(P(\pi))$ and $Q(\pi^{rc}) = \mathrm{ev}(Q(\pi))$. In particular, $\pi^{rc} = \pi$ if and only if $P(\pi)$ and $Q(\pi)$ are self-evacuating.*

Proof. This result is part of an exercise [12, p. 136], but we include a proof here for completeness.

One finds in Sagan's text [12, Thm. 3.2.3, Thm. 3.6.6, Thm. 3.9.4] that for all $\pi \in S_n$, we have

$$P(\pi^r) = P(\pi)^t, \tag{4}$$

where $P(\pi)^t$ is the transpose of $P(\pi)$,

$$P(\pi^i) = Q(\pi), \tag{5}$$

$$Q(\pi^i) = P(\pi), \tag{6}$$

and

$$Q(\pi^r) = \mathrm{ev}(Q(\pi)^t). \tag{7}$$

Combining these we find

$$\begin{aligned}
P(\pi^{rc}) &= P(\pi^{riic}) && \text{(since } i^2 \text{ is the identity)}\\
&= P(\pi^{riri}) && \text{(since } ic = ri)\\
&= Q(\pi^{rir}) && \text{(by (5))}\\
&= \mathrm{ev}(Q(\pi^{ri})^t) && \text{(by (7))}\\
&= \mathrm{ev}(P(\pi^{r})^t) && \text{(by (6))}\\
&= \mathrm{ev}(P(\pi)) && \text{(by (4))}.
\end{aligned}$$

The proof that $Q(\pi^{rc}) = \mathrm{ev}(Q(\pi))$ is similar.

The last statement of the theorem is immediate from the first statement, since the RSK correspondence is a bijection. $\square$

4 Self-Evacuating Tableaux and Domino Tableaux

In view of Theorem 3.8, we can enumerate $S_n^{rc}(k \dots 21)$ by enumerating ordered pairs of self-evacuating tableaux of the same shape. In this section we show that self-evacuating tableaux are in bijection with slightly simpler objects called domino tableaux. As in the previous section, we follow the approach of van Leeuwen [9]. We begin by considering a simpler way of producing a smaller standard tableau from a given standard tableau.

Definition 4.1. *Fix $n \geq 1$. For any standard tableau P with n boxes, we write $\mathrm{d}(P)$ to denote the standard tableau with $n-1$ boxes obtained by removing n and its box from P.*

The inverse of d is equally easy to describe.

Definition 4.2. *Fix $n \geq 0$. For any standard tableau P with n boxes and any box c not in P for which $P \cup \{c\}$ is a legal Ferrers diagram, we write $\mathrm{a}_c(P)$ to denote the standard tableau with $n+1$ boxes obtained by putting $n+1$ in c.*

Proposition 4.3. *Fix $n \geq 1$. If P is a standard tableau with n boxes, then the following hold.*

(i) *If c is the box removed from P in creating $\mathrm{d}(P)$ then $\mathrm{a}_c(\mathrm{d}(P)) = P$.*

(ii) *If c is a box not in P for which $P \cup \{c\}$ is a legal Ferrers diagram then $\mathrm{d}(\mathrm{a}_c(P)) = P$.*

Proof. These are immediate from Definitions 4.1 and 4.2. $\square$

It is routine to prove our next result, which says that e commutes with d and i_c commutes with a_c, so we omit the details.

Proposition 4.4. *Fix $n \geq 2$. If P is a standard tableau with n boxes, then the following hold.*

(i) $\mathrm{d}(\mathrm{e}(P)) = \mathrm{e}(\mathrm{d}(P))$.

(ii) *If c_1 and c_2 are distinct, adjacent boxes not in P for which $P \cup \{c_1\}$ and $P \cup \{c_1, c_2\}$ are legal Ferrers diagrams then $\mathrm{a}_{c_2}(\mathrm{i}_{c_1}(P)) = \mathrm{i}_{c_2}(\mathrm{a}_{c_1}(P))$.*

As we show next, e and d are dual with respect to evacuation, as are i_c and a_c.

Proposition 4.5. *Fix $n \geq 1$. If P is a standard tableau with n boxes, then the following hold.*

(i) $\mathrm{ev}(\mathrm{e}(P)) = \mathrm{d}(\mathrm{ev}(P))$.

(ii) $\mathrm{ev}(\mathrm{d}(P)) = \mathrm{e}(\mathrm{ev}(P))$.

(iii) *If c is a box not in P for which $P \cup \{c\}$ is a legal Ferrers diagram then $\mathrm{ev}(\mathrm{i}_c(P)) = \mathrm{a}_c(\mathrm{ev}(P))$.*

(iv) *If c is a box not in P for which $P \cup \{c\}$ is a legal Ferrers diagram then $\mathrm{ev}(\mathrm{a}_c(P)) = \mathrm{i}_c(\mathrm{ev}(P))$.*

Proof. (i) By definition $\mathrm{ev}(P)$ records the order in which boxes are removed from P when e is applied n times, in such a way that the n in $\mathrm{ev}(P)$ is contained in the first box removed. Thus, $\mathrm{ev}(\mathrm{e}(P))$ records the order in which boxes are removed from $\mathrm{e}(P)$, which must be $\mathrm{d}(\mathrm{ev}(P))$, as claimed.

(ii) In (i), replace P with $\mathrm{ev}(P)$, apply ev to both sides of the result, and use the fact [9, Thm. 2.2.1] that $\mathrm{ev}(\mathrm{ev}(P)) = P$.

(iii) In view of Proposition 3.5(ii), the first box ev removes from $\mathrm{i}_c(P)$ is c, and the result follows.

(iv) This is similar to the proof of (ii), using (iii). □

The dualities of e and d, and of i_c and a_c, allow us to describe how to build self-evacuating tableaux two boxes at a time.

Proposition 4.6. *Fix $n \geq 2$ and suppose P is a self-evacuating tableau with n boxes. Then the following hold.*

(i) *The box* e *removes from P and the box* d *removes from* $\mathrm{e}(P)$ *are adjacent, and* $\mathrm{d}(\mathrm{e}(P))$ *is a self-evacuating tableau.*

(ii) *If c_1 and c_2 are distinct, adjacent boxes not in P for which $P \cup \{c_1\}$ and $P \cup \{c_1, c_2\}$ are both legal Ferrers diagrams then $\mathrm{a}_{c_2}(\mathrm{i}_{c_1}(P))$ is a self-evacuating tableau.*

Proof. (i) Since P is self-evacuating, the box e removes from P is the box containing n. Therefore n slides to an adjacent box when e is applied, and is then removed from that adjacent box (as $n-1$) by d.

To see that $\mathrm{d}(\mathrm{e}(P))$ is self-evacuating, note that

$$\begin{aligned}
\mathrm{ev}(\mathrm{d}(\mathrm{e}(P))) &= \mathrm{e}(\mathrm{ev}(\mathrm{e}(P))) && \text{(by Proposition 4.5(ii))}\\
&= \mathrm{e}(\mathrm{d}(\mathrm{ev}(P))) && \text{(by Proposition 4.5(i))}\\
&= \mathrm{e}(\mathrm{d}(P)) && \text{(since } P \text{ is self-evacuating)}\\
&= \mathrm{d}(\mathrm{e}(P)) && \text{(by Proposition 4.4(i)).}
\end{aligned}$$

(ii) This is similar to the last part of the proof of (i), using Propositions 4.4(ii) and 4.5(iii),(iv). □

Proposition 4.6 suggests that the entries in a self-evacuating tableau come naturally in adjacent pairs. To make this idea precise, we recall the notion of a domino tableau.

Definition 4.7. *Suppose $n \geq 1$ and $\lambda \vdash n$. A* domino tableau *of shape λ is a filling of the Ferrers diagram of λ with $1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ for which the following hold.*

(i) *Each of $1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ appears exactly twice.*

(ii) *The entries in each row are nondecreasing from left to right.*

(iii) *The entries in each column are nondecreasing from top to bottom.*

(iv) *For each i, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, the two boxes containing i are adjacent.*

(v) *If n is odd then the box in the upper left corner of the Ferrers diagram of λ has no entry.*

As we describe next, domino tableaux of a given shape are in bijection with self-evacuating tableaux of that shape. To give one such bijection, suppose $\lambda \vdash 2n$ and D is a domino tableau of shape λ. (The construction for $\lambda \vdash 2n+1$ is similar.) Let P_1 be a Ferrers diagram with two empty boxes, in the same shape as the two boxes in D which contain 1. Place a 1 in the box in the upper left corner of this diagram, and a 2 in the other box. We now build a sequence $P_1, P_2, \ldots, P_n$ of standard tableaux, in which P_k contains $1, 2, \ldots, 2k$ and the shape of P_k is the shape of the set of boxes in D which contain $1, 2, \ldots, k$. To construct P_k from P_{k-1}, first let c_1 and c_2 be the boxes which contain k in D, chosen so that c_1 is either left of or above c_2. Then set $P_k = \mathrm{a}_{c_2}(\mathrm{i}_{c_1}(P_{k-1}))$. The last tableau in our sequence is the image of our map.

Definition 4.8. *If D is a domino tableau of shape $\lambda \vdash n$, then we write $\varphi(D)$ to denote the tableau $P_{\lfloor \frac{n}{2} \rfloor}$ obtained from the construction described in the paragraph above.*

Example 4.9. *If D is the domino tableau given below on the left, then $\varphi(D)$ is the standard tableau given below on the right.*

1	1	3	6	6
2	2	3		
4	5	5		
4				

1	2	3	7	12
4	5	9		
6	8	11		
10				

Example 4.10. *If D is the domino tableau given below on the left, then $\varphi(D)$ is the standard tableau given below on the right.*

	1	1	5
2	2	4	5
3	3	4	
6	6		

1	3	4	8
2	7	9	12
5	10	11	
6	13		

Before showing that φ is a bijection, we should show that its image is contained in the set of self-evacuating tableaux.

Proposition 4.11. *If D is a domino tableau of shape $\lambda \vdash n$, then $\varphi(D)$ is self-evacuating.*

Proof. Let $P_1, \ldots, P_{\lfloor \frac{n}{2} \rfloor} = \varphi(D)$ be the sequence of tableaux obtained in the construction of $\varphi(D)$. Clearly P_1 is self-evacuating, so by Proposition 4.6(ii) and induction on k, the tableau P_k is self-evacuating for all k, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. In particular, $\varphi(D) = P_{\lfloor \frac{n}{2} \rfloor}$ is self-evacuating, as desired. $\square$

To show φ is a bijection, we describe φ^{-1}. To do this, suppose P is a self-evacuating tableau with $n \geq 1$ boxes, and let Q be a copy of the Ferrers diagram of λ in which every box is empty. Place $\lfloor \frac{n}{2} \rfloor$ in each of the two adjacent boxes of Q which are in P but not in $\mathrm{d}(\mathrm{e}(P))$. By Proposition 4.6(i), the tableau $\mathrm{d}(\mathrm{e}(P))$ is self-evacuating, so we may repeat this process until we obtain a domino tableau.

Definition 4.12. *If P is a self-evacuating tableau of shape $\lambda \vdash n$, then we write $\psi(P)$ to denote the domino tableau obtained from the construction described in the paragraph above.*

In view of Propositions 3.5, 4.3, and 4.4, the map ψ is a step by step reversal of φ, so φ and ψ are inverse functions. In particular, by induction on n we have the following result.

Theorem 4.13. *The following hold for all $n \geq 1$.*

(i) *For any domino tableau D of shape $\lambda \vdash n$, we have $\psi(\varphi(D)) = D$.*

(ii) *For any self-evacuating tableau P of shape $\lambda \vdash n$, we have $\varphi(\psi(P)) = P$.*

In particular, for all $n \geq 1$, the maps φ and ψ are inverse bijections between the set of all domino tableaux of shape $\lambda \vdash n$ and the set of all self-evacuating tableaux of shape $\lambda \vdash n$.

5 Domino Tableaux and Pairs of Tableaux

We have now reduced the problem of enumerating $S_n^{rc}(k \dots 21)$ to the problem of enumerating ordered pairs of domino tableaux of the same shape with n boxes and at most $k-1$ rows. In this section we show how such a domino tableau may be uniquely factored into an ordered pair of tableaux. Although the factorization we give can be described more generally (as van Leeuwen [9] does), throughout we consider domino tableaux with exactly $k-1$ rows, and we include domino tableaux with fewer rows in this set by appending rows of length 0.

Suppose D is a domino tableau with n boxes and exactly $k-1$ rows. To describe how to factor D into an ordered pair of tableaux, we first consider the boundary of D. Beginning at the southwest corner of this boundary, encode the southeast boundary as an infinite sequence of vertical and horizontal steps, which contains exactly $k-1$ vertical steps. For instance, if $k = 6$ and the shape of D is $7, 3, 2$ then the corresponding string of steps is $vvhhvhvhhhhvhhhh\dots$. Number these steps from left to right beginning with 1, let $P_o(D)$ be the Ferrers diagram whose southeast boundary is given by the steps with odd numbers, and let $P_e(D)$ be the Ferrers diagram whose southeast boundary is given by the steps with even numbers. For instance, if $k = 6$ and the shape of D is $7, 3, 2$ then $P_o(D)$ has shape $1, 1, 0$ and $P_e(D)$ has shape $4, 0$.

To fill in the entries of $P_o(D)$ and $P_e(D)$, we imagine building D one domino at a time, in increasing order of its entries. Each domino changes three consecutive entries of the boundary string of D from vvh to hvv, or from vhh to hhv. Therefore, each domino adds either one box to $P_o(D)$ or one box to $P_e(D)$; put the label of the domino in that box of $P_o(D)$ or $P_e(D)$.

Definition 5.1. *For any domino tableau D, we write $(P_o(D), P_e(D))$ to denote the ordered pair of tableaux obtained from D by the process described above.*

Example 5.2. *If D is the domino tableau below on the left and $k = 4$, then $P_o(D)$ is the tableau below in the center and $P_e(D)$ is the tableau below on the right.*

1	2	2	4	4	8	9
1	3	3	6	6	8	9
5	5	7	7			

3	6	8
5	7	

1	2	4	9

As we show next, domino tableaux with $2n$ boxes and exactly $k-1$ rows can only factor into certain pairs of tableaux.

Proposition 5.3. *Fix $n \geq 1$ and $k \geq 2$, and suppose D is a domino tableau with $2n$ boxes and exactly $k-1$ rows. Then the following hold.*

(i) *Each of $1, 2, \dots, n$ appears in exactly one of $P_o(D), P_e(D)$.*

(ii) *$P_o(D)$ has exactly $\lceil \frac{k-1}{2} \rceil$ rows and $P_e(D)$ has exactly $\lfloor \frac{k-1}{2} \rfloor$ rows. In each case some of these rows may have length 0.*

Proof. (i) This is immediate from the construction of $P_o(D)$ and $P_e(D)$.

(ii) This is immediate by induction on n. $\square$

We have a similar result for domino tableaux with $2n+1$ boxes.

Proposition 5.4. *Fix $n \geq 1$ and $k \geq 2$, and suppose D is a domino tableau with $2n+1$ boxes and exactly $k-1$ rows. Then the following hold.*

(i) *Each of $1, 2, \ldots, n$ appears in exactly one of $P_o(D), P_e(D)$.*

(ii) *If k is odd then $P_o(D)$ has exactly $\frac{k+1}{2}$ rows and $P_e(D)$ has exactly $\frac{k-3}{2}$ rows. In each case some of these rows may have length 0.*

(iii) *If k is even then $P_o(D)$ has exactly $\frac{k}{2} - 1$ rows and $P_e(D)$ has exactly $\frac{k}{2}$ rows. In each case some of these rows may have length 0.*

Proof. This is similar to the proof of Proposition 5.3. $\square$

We now show that the map $D \mapsto (P_o(D), P_e(D))$ is a bijection between domino tableaux with $2n$ boxes and $k-1$ rows and ordered pairs (P_o, P_e) of tableaux which satisfy Proposition 5.3(i),(ii).

Theorem 5.5. *Fix $n \geq 1$ and $k \geq 2$, and suppose (P_o, P_e) is an ordered pair of tableaux which satisfies Proposition 5.3(i),(ii) (resp. Proposition 5.4(i),(ii),(iii)). Then there is a unique domino tableau D with $2n$ (resp. $2n+1$) boxes and exactly $k-1$ rows for which $P_o(D) = P_o$ and $P_e(D) = P_e$.*

Proof. To recover the shape of a domino tableau D which maps to (P_o, P_e), first encode the boundaries of P_o and P_e as strings of hs and vs as in the original map. Then there is a unique string of hs and vs with P_o's boundary string in its even positions and P_e's boundary string in its odd positions; this must be the boundary string of D. Since P_o has $\lceil \frac{k-1}{2} \rceil$ rows and P_e has $\lfloor \frac{k-1}{2} \rfloor$ rows, D will have $k-1$ rows.

To recover the entries in the boxes of D, first note that if D has just two boxes then each must contain a 1. If D has $2n > 2$ boxes then we first label two adjacent boxes in D with n, and proceed by induction.

Since P_o and P_e satisfy Proposition 5.3(i), there is a unique box in P_o or P_e which contains n. This box is the intersection of a column created by an h and a row created by a v in the boundary string of its shape. This h and v correspond to an h and a v in the boundary string of D, so this box has a unique corresponding box in D; by construction of $P_o(D)$ and $P_e(D)$ we must have an m in this box. To find a second box in D in which to place an n, first note that the n in P_o or P_e is in an outer corner, which is formed by an hv in the boundary string of its shape. Therefore the n in D is in a box formed by a substring of the form hhv or hvv. In either case, the box containing n is adjacent to exactly one outer corner of D, so this outer corner must also contain an n. Now remove these two boxes from D and remove the box containing n from P_o or P_e. The resulting D is the partial domino tableau obtained from the resulting P_o and P_e, so by induction it has a unique extension to a domino tableau. $\square$

6 Enumerations of Permutations

We now have the tools we need to enumerate $S_n^{rc}(k\dots21)$ in terms of classical pattern-avoiding permutations.

Theorem 6.1. *For all $n \geq 0$ and all $k \geq 2$ we have*

$$|S_{2n}^{rc}(k\dots21)| = \sum_{j=0}^{n} \binom{n}{j}^2 \left|S_j\left(\left\lceil\frac{k+1}{2}\right\rceil\dots21\right)\right| \left|S_{n-j}\left(\left\lfloor\frac{k+1}{2}\right\rfloor\dots21\right)\right|. \tag{8}$$

Proof. By Theorems 2.3, 2.4, 3.8, 4.13, and 5.5, the set $S_{2n}(k\dots21)$ is in bijection with the set of ordered quadruples (P_o, P_e, Q_o, Q_e) of tableaux in which the ordered pairs (P_o, P_e) and (Q_o, Q_e) satisfy Proposition 5.3(i),(ii), the tableaux P_o and Q_o have the same shape, and the tableaux P_e and Q_e have the same shape. Now let j be the number of boxes in P_o (and thus in Q_o) and observe that $0 \leq j \leq n$. For each ordered pair (P_o, Q_o) (resp. (P_e, Q_e)), let P' (resp. Q') be the set of entries in P_o (resp. Q_o), replace each tableau with the standard tableau of the same shape whose entries are in the same relative positions, and apply the inverse of the RSK correspondence. This gives a bijection between the set of ordered quadruples (P_o, P_e, Q_o, Q_e) and the set of ordered quadruples (P', Q', π_o, π_e), where P' and Q' are subsets of $[n]$ of size j, $\pi_o \in S_j\left(\left\lceil\frac{k+1}{2}\right\rceil\dots21\right)$, and $\pi_e \in S_{n-j}\left(\left\lfloor\frac{k+1}{2}\right\rfloor\dots21\right)$. These quadruples are enumerated by the expression on the right side of (8). □

Theorem 6.2. *For all $n \geq 0$ and all $k \geq 2$ the following hold.*

(i) *If k is odd then*

$$|S_{2n+1}^{rc}(k\dots21)| = \sum_{j=0}^{n} \binom{n}{j}^2 \left|S_j\left(\frac{k+3}{2}\dots21\right)\right| \left|S_{n-j}\left(\frac{k-1}{2}\dots21\right)\right|. \tag{9}$$

(ii) *If k is even then*

$$|S_{2n+1}^{rc}(k\dots21)| = \sum_{j=0}^{n} \binom{n}{j}^2 \left|S_j\left(\frac{k}{2}\dots21\right)\right| \left|S_{n-j}\left(\frac{k+2}{2}\dots21\right)\right|. \tag{10}$$

Proof. This is similar to the proof of Theorem 6.1. □

It is easy to see that

$$|S_n(21)| = 1 \qquad (n \geq 0), \tag{11}$$

and it is well-known that

$$|S_n(321)| = C_n \qquad (n \geq 0). \tag{12}$$

In addition, Gessel [6], Gessel, Weinstein, and Wilf [5], and Bousquet-Mélou [2] have shown that

$$|S_n(4321)| = \frac{1}{(n+1)^2(n+2)} \sum_{j=0}^{n} \binom{2j}{j}\binom{n+1}{j+1}\binom{n+2}{j+1} \qquad (n \geq 0). \tag{13}$$

Combining these results with Theorems 6.1 and 6.2 gives us the following explicit enumerations.

Corollary 6.3. *For all $n \geq 0$ we have*

$$|S^{rc}_{2n}(321)| = \binom{2n}{n}, \tag{14}$$

$$|S^{rc}_{2n+1}(4321)| = |S^{rc}_{2n}(4321)| = \sum_{j=0}^{n} \binom{n}{j}^2 C_j,$$

$$|S^{rc}_{2n}(54321)| = \sum_{j=0}^{n} \binom{n}{j}^2 C_j C_{n-j},$$

$$|S^{rc}_{2n+1}(54321)| = \sum_{j=0}^{n} \binom{n}{j}^2 \frac{1}{(j+1)^2(j+2)} \sum_{i=0}^{j} \binom{2i}{i}\binom{j+1}{i+1}\binom{j+2}{i+1},$$

$$|S^{rc}_{2n+1}(654321)| = |S^{rc}_{2n}(654321)| = \sum_{j=0}^{n} \binom{n}{j}^2 \frac{C_{n-j}}{(j+1)^2(j+2)} \sum_{i=0}^{j} \binom{2i}{i}\binom{j+1}{i+1}\binom{j+2}{i+1}.$$

Proof. To prove (14), set $k = 3$ in (8) and use (11) and the fact that $\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$ to simplify the result. The proofs of the remaining results are similar, using (12) and (13). □

We note that Egge [4] has previously proved (14) using generating functions and the kernel method.

7 Enumerations of Involutions

Having enumerated $S^{rc}_n(k \ldots 21)$, we now turn our attention to $I^{rc}_n(k \ldots 21)$. We begin with the following characterization of $P(\pi)$ and $Q(\pi)$ when π is an involution.

Proposition 7.1. *([12, Thm. 3.6.6]) Fix $n \geq 1$ and suppose $\pi \in S_n$. Then π is an involution if and only if $P(\pi) = Q(\pi)$.*

Proposition 7.1 allows us to prove the following analogue of Theorems 6.1 and 6.2 for involutions.

Theorem 7.2. *The following hold for all $n \geq 0$ and all $k \geq 2$.*

(i) *We have*

$$|I^{rc}_{2n}(k \ldots 21)| = \sum_{j=0}^{n} \binom{n}{j} \left| I_j \left(\left\lceil \frac{k+1}{2} \right\rceil \ldots 21 \right) \right| \left| I_{n-j} \left(\left\lfloor \frac{k+1}{2} \right\rfloor \ldots 21 \right) \right|. \tag{15}$$

(ii) *If k is odd then*

$$|I^{rc}_{2n+1}(k \ldots 21)| = \sum_{j=0}^{n} \binom{n}{j} \left| I_j \left(\frac{k+3}{2} \ldots 21 \right) \right| \left| I_{n-j} \left(\frac{k-1}{2} \ldots 21 \right) \right|. \tag{16}$$

(iii) *If k is even then*

$$|I_{2n+1}^{rc}(k\dots 21)| = \sum_{j=0}^{n} \binom{n}{j} \left| I_j\left(\frac{k}{2}\dots 21\right)\right| \left| I_{n-j}\left(\frac{k+2}{2}\dots 21\right)\right|. \tag{17}$$

Proof. This is similar to the proofs of Theorems 6.1 and 6.2, using Proposition 7.1. □

It is easy to see that

$$|I_n(21)| = 1 \qquad (n \geq 0), \tag{18}$$

Gessel [6] and Simion and Schmidt [15] have shown that

$$|I_n(321)| = \binom{n}{\lfloor \frac{n}{2} \rfloor} \qquad (n \geq 0), \tag{19}$$

and Gessel [6] has shown that

$$|I_n(4321)| = M_n \qquad (n \geq 0) \tag{20}$$

and

$$|I_n(54321)| = C_{\lfloor \frac{n+1}{2} \rfloor} C_{\lceil \frac{n+1}{2} \rceil} \qquad (n \geq 0). \tag{21}$$

Here M_n is the nth Motzkin number [3], which may be defined by $M_0 = 1$ and $M_n = M_{n-1} + \sum_{k=2}^{n} M_{k-2} M_{n-k}$ for $n \geq 1$. Combining these results with Theorem 7.2 gives us the following explicit enumerations.

Corollary 7.3. *For all $n \geq 0$ we have*

$$|I_{2n}^{rc}(321)| = 2^n, \tag{22}$$

$$|I_{2n+1}^{rc}(4321)| = |I_{2n}^{rc}(4321)| = \sum_{j=0}^{n} \binom{n}{j} \binom{j}{\lfloor \frac{j}{2} \rfloor},$$

$$|I_{2n}^{rc}(54321)| = \binom{n}{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{\lfloor \frac{n+1}{2} \rfloor}, \tag{23}$$

$$|I_{2n+1}^{rc}(54321)| = C_{n+1}, \tag{24}$$

$$|I_{2n+1}^{rc}(654321)| = |I_{2n}^{rc}(654321)| = \sum_{j=0}^{n} \binom{n}{j} \binom{n-j}{\lfloor \frac{n-j}{2} \rfloor} M_j,$$

$$|I_{2n}^{rc}(7654321)| = \sum_{j=0}^{n} \binom{n}{j} M_j M_{n-j},$$

$$|I_{2n+1}^{rc}(7654321)| = \sum_{j=0}^{n} \binom{n}{j} \binom{n-j}{\lfloor \frac{n-j}{2} \rfloor} C_{\lfloor \frac{j+1}{2} \rfloor} C_{\lceil \frac{j+1}{2} \rceil},$$

and

$$|I_{2n+1}^{rc}(87654321)| = |I_{2n}^{rc}(87654321)| = \sum_{j=0}^{n} \binom{n}{j} C_{\lfloor \frac{j+1}{2} \rfloor} C_{\lceil \frac{j+1}{2} \rceil} M_{n-j}.$$

Proof. To prove (24), set $k = 5$ in (16) and use (18) and (20) to simplify the result, obtaining

$$|I_{2n+1}^{rc}(54321)| = \sum_{j=0}^{n} \binom{n}{j} M_j \qquad (n \geq 0).$$

It is well-known that $\sum_{j=0}^{\infty} M_j x^j = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$ and $\sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$, so we can use Wilf's snake oil method [16] to find

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} M_j x^n &= \sum_{j=0}^{\infty} M_j x^j \sum_{n=0}^{\infty} \binom{j+n}{j} x^n \\ &= \frac{1}{1-x} \sum_{j=0}^{\infty} M_j \left(\frac{x}{1-x} \right)^j \\ &= \frac{1-2x-\sqrt{1-4x}}{2x^2} \\ &= \sum_{n=0}^{\infty} C_{n+1} x^n. \end{aligned}$$

It follows that $\sum_{j=0}^{n} \binom{n}{j} M_j = C_{n+1}$ for $n \geq 0$, as desired.

The proofs of the remaining results are similar. In the proof of (23) one can use the Chu-Vandermonde identity [1, Cor. 2.2.3 and Sec. 2.7] to show that

$$\sum_{j=0}^{n} \binom{n}{j} \binom{j}{\lfloor \frac{j}{2} \rfloor} \binom{n-j}{\lfloor \frac{n-j}{2} \rfloor} = \binom{n}{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{\lfloor \frac{n+1}{2} \rfloor} \qquad (n \geq 0).$$

□

We note that Egge [4] has previously proved (22) using combinatorial methods.

Acknowledgment

This work was completed while the author was on sabbatical at the University of Pennsylvania. The author thanks the University of Pennsylvania Mathematics Department, and especially Jim Haglund, for their support and hospitality. The author also thanks the referee for her/his valuable suggestions.

References

[1] G. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1999.

[2] M. Bousquet-Mélou. Four classes of pattern-avoiding permutations under one roof: Generating trees with two labels. *Electron. J. Combin.*, 9(2):#R19, 2003.

[3] R. Donaghey and L. W. Shapiro. Motzkin numbers. *J. Combin. Theory, Series A*, 23(3):291–301, 1977.

[4] E. S. Egge. Restricted symmetric permutations. *Ann. Comb.*, 11:405–434, 2007.

[5] I. Gessel, J. Weinstein, and H. Wilf. Lattice walks in $\mathbf{Z}^d$ and permutations with no long ascending subsequences. *Electron. J. Combin.*, 5(1):#R2, 1998.

[6] I. M. Gessel. Symmetric functions and P-recursiveness. *J. Combin. Theory, Series A*, 53:257–285, 1990.

[7] C. Greene. An extension of Schensted's theorem. *Adv. in Math.*, 14:254–265, 1974.

[8] D. E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.*, 34:709–727, 1970.

[9] M. van Leeuwen. The Robinson-Schensted and Schützenberger algorithms, an elementary approach. *Electron. J. Combin.*, 3(2):#R15, 1996.

[10] D. Lonoff and J. Ostroff. Symmetric permutations avoiding a pattern of length three and a pattern of length four. preprint.

[11] G. de B. Robinson. On representations of the symmetric group. *Amer. J. Math*, 60:745–760, 1938.

[12] B. E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. Springer-Verlag, 2nd edition, 2000.

[13] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.

[14] M. P. Schützenberger. La correspondence de Robinson. In D. Foata, editor, *Combinatoire et Représentation du Groupe Symétrique*, volume 579 of *Lecture Notes in Math.*, pages 59–135, 1977.

[15] R. Simion and F. Schmidt. Restricted permutations. *Europ. J. Combin.*, 6:383–406, 1985.

[16] H. Wilf. *generatingfunctionology*. A. K. Peters, 3rd edition, 2005.