From quasisymmetric expansions to Schur expansions via a modified inverse Kostka matrix*

Eric Egge
Carleton College
Northfield, MN 55057
eegge@carleton.edu

Nicholas A. Loehr
Virginia Tech
Blacksburg, VA 24061
nloehr@vt.edu

Gregory S. Warrington
University of Vermont
Burlington, VT 05401
gwarring@cems.uvm.edu

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Corresponding author:
Gregory S. Warrington
Dept. of Mathematics and Statistics
University of Vermont
16 Colchester Ave.
Burlington, VT 05401
gwarring@cems.uvm.edu

Abstract

Every symmetric function f can be written uniquely as a linear combination of Schur functions, say $f = \sum_\lambda x_\lambda s_\lambda$, and also as a linear combination of fundamental quasisymmetric functions, say $f = \sum_\alpha y_\alpha Q_\alpha$. For many choices of f arising in the theory of Macdonald polynomials and related areas, one knows the quasisymmetric coefficients y_α and wishes to compute the Schur coefficients x_λ. This paper gives a general combinatorial formula expressing each x_λ as a linear combination of the y_α's, where each coefficient in this linear combination is $+1$, -1, or 0. This formula arises by suitably modifying Eğecioğlu and Remmel's combinatorial interpretation of the inverse Kostka matrix involving special rim-hook tableaux.

1 Introduction

The Schur symmetric functions s_λ are inextricably linked with the representations of, and geometry related to, the symmetric group and general linear group. One manifestation of this connection is that the coefficients in the Schur expansion of a symmetric function frequently carry important data. For example, the Littlewood-Richardson coefficients in the Schur expansion of $s_\lambda s_\mu$ encode multiplicities of irreducible representations inside certain tensor product representations.

A fundamental problem in the theory of symmetric functions is to express a given symmetric function f as an explicit linear combination of Schur functions. If the monomial symmetric function expansion of f is known, one can use the inverse of the classical Kostka matrix to change bases. In fact, Eğecioğlu and Remmel give a combinatorial interpretation for the inverse Kostka matrix in which each entry is expressed as a signed sum of *special rim-hook tableaux* [5]. Another interpretation of the inverse Kostka matrix is given by Duan [4]. The advantages of having an explicit combinatorial interpretation are illustrated, for example, in Yang and Remmel [11] and Vallejo [13].

In the last few years, many important examples have arisen in which f has a natural combinatorial formula when written as a linear combination of Gessel's fundamental quasisymmetric functions Q_α [6]. For instance, such formulas exist when f is a Lascoux-Leclerc-Thibon (LLT) symmetric polynomial [8], when f is a modified Macdonald polynomial [8], and (conjecturally) when f is the image of a Schur function under the Bergeron-Garsia nabla operator [10]. This suggests the following general linear-algebraic question. Suppose we are given *any* symmetric function f and the coefficients y_α in the fundamental quasisymmetric expansion $f = \sum_\alpha y_\alpha Q_\alpha$. How can we use the y_α to find the coefficients x_λ in the Schur expansion $f = \sum_\lambda x_\lambda s_\lambda$? The main result of this paper, Theorem 11, is an explicit combinatorial formula

$$x_\lambda = \sum_\alpha y_\alpha K^*(\alpha, \lambda), \tag{1}$$

where each $K^*(\alpha, \lambda) \in \{-1, 0, +1\}$. In fact, K^* is the matrix of a certain *projection* of the space of quasisymmetric functions onto the subspace of symmetric functions. The "transition matrix" K^* arises by suitably modifying the inverse Kostka matrix of Eğecioğlu and Remmel. We will see that the α, λ-entry of K^* is nonzero when there exists a (necessarily unique) "flat" special rim-hook tableau of shape λ and content α.

The matrix K^* is relatively sparse and has no nonzero entries other than ± 1. In addition, its combinatorial description requires fewer objects than that of the inverse Kostka matrix (see

Table 1). The passage from quasisymmetric functions to Schur functions involves (unavoidably) the use of both positive and negative coefficients, as seen in (1), but the amount of cancellation involved is less than what is needed when starting from a monomial expansion. Thus, it may be easier to define involutions that cancel out negative terms in the quasisymmetric-to-Schur case than it has been for the monomial-to-Schur case.

A different approach to this change-of-basis problem has been developed by Assaf [1, 2]. For certain quasisymmetric expansions where the y_α's satisfy additional structural axioms, she constructs *D-graphs* which, through a recursive algorithm, can be converted to other graphs whose connected components correspond to terms in the Schur expansion.

The rest of the paper is organized as follows. Section 2 reviews the facts we need about Schur functions and fundamental quasisymmetric functions. Sections 3 and 4 discuss square and rectangular versions of the Kostka matrix and its (right) inverse. Section 5 adjusts these matrices by an inclusion-exclusion factor, which leads to the proof of the desired result (1) in Section 6. We derive a combinatorial interpretation of the matrix K^* in Section 7. Theorem 19 in Section 8 gives an explicit formula for the number of flat special rim-hook tableaux of a given shape while Theorem 21 offers a generating function for such objects. The paper concludes in Section 9 with a typical application of our results to the theory of Macdonald polynomials.

2 The Bases s_λ and Q_α

For each $n \geq 1$, let $p(n)$ be the number of integer partitions of n, and let $c(n) = 2^{n-1}$ be the number of compositions of n. The notation $\lambda \vdash n$ means that λ is a partition of n, and $\alpha \models n$ means that α is a composition of n. For any logical statement P, let $\chi(P) = 1$ if P is true, and $\chi(P) = 0$ if P is false.

Our desired formula (1) is a consequence of the linear-algebraic relationship between Schur functions and fundamental quasisymmetric functions. So, we shall only require the following three well-known facts about these objects. First, there is a vector space Sym_n of dimension $p(n)$ with a basis $\{s_\lambda : \lambda \vdash n\}$ indexed by partitions λ of n; s_λ is called a *Schur symmetric function*. Second, there is a vector space QSym_n of dimension $c(n)$ with a basis $\{Q_\alpha : \alpha \models n\}$ indexed by compositions α of n; Q_α is called a *fundamental quasisymmetric function*. Third, Sym_n is a subspace of QSym_n, and there is the following combinatorial rule for expressing Schur functions in terms of fundamental quasisymmetric functions:

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} Q_{\mathrm{Des}'(T)}. \tag{2}$$

Here, $\mathrm{SYT}(\lambda)$ is the set of *standard tableaux of shape* λ. Denoting the Ferrers diagram of λ by $\mathrm{dg}(\lambda)$, these standard tableaux can be defined as those bijections $T : \mathrm{dg}(\lambda) \to \{1, 2, \ldots, n\}$ for which T is increasing along the rows and up the columns of $\mathrm{dg}(\lambda)$. (We draw diagrams with the longer rows at the bottom.) Given $T \in \mathrm{SYT}(\lambda)$, define the *reading word* $\mathrm{rw}(T)$ by scanning the rows of T from left to right, starting at the shortest row. Then define $\mathrm{Des}(T)$ to be the set of $i \in \{1, 2, \ldots, n-1\}$ such that $i+1$ appears to the left of i in $\mathrm{rw}(T)$. If $\mathrm{Des}(T) = \{i_1 < i_2 < \cdots < i_k\}$, define $\mathrm{Des}'(T)$ to be the composition $(i_1, i_2 - i_1, i_3 - i_2, \ldots, n - i_k) \models n$. Also, define a *horizontal strip* of T to be a collection of cells in the underlying Ferrers diagram, no two of which are in the same column. One can check that $\mathrm{Des}'(T) = (\alpha_1, \ldots, \alpha_s)$ iff T can be decomposed into a succession of horizontal strips $H_1, \ldots, H_s$ such that: $|H_i| = \alpha_i$, the entries of

H_i form a maximal subsequence of numerically consecutive entries of $\mathrm{rw}(T)$, and $i < j$ implies all labels in H_i are less than all labels in H_j. For example,

$$T = \begin{array}{|c|c|cc} \hline 8 & 9 \\ \hline 3 & 5 & \multicolumn{1}{c|}{7} \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

is in $\mathrm{SYT}(4,3,2)$ and has $\mathrm{rw}(T) = 893571246$, $\mathrm{Des}(T) = \{2,4,6,7\}$, and $\mathrm{Des}'(T) = (2,2,2,1,2)$.

Remark 1. One may derive (2) from the combinatorial definitions of Schur polynomials and fundamental quasisymmetric polynomials (which we have not stated here) using a *standardization bijection*. The bijection maps a semistandard tableau counted by s_λ to the pair consisting of its standardization $T \in \mathrm{SYT}(\lambda)$ (see Section 3) and its content word, which is one of the monomials in $Q_{\mathrm{Des}'(T)}$; see [9, Thm. 18] for more details. (A similar idea is used to prove Lemma 9 below.) An alternative proof of (2), based on posets, can be found in [12, Ch. 7].

3 The Kostka Matrix

Here and below, we will view the Kostka matrix K_n as a *rectangular* matrix of order $p(n) \times c(n)$, with rows indexed by partitions of n and columns indexed by compositions of n. The general entry of this matrix is

$$K_n(\lambda, \alpha) = |\,\mathrm{SSYT}(\lambda, \alpha)| \qquad (\lambda \vdash n, \alpha \models n)$$

where $\mathrm{SSYT}(\lambda, \alpha)$ is the set of *semistandard tableaux* of shape λ and type α. Such a tableau is a map $T : \mathrm{dg}(\lambda) \to \{1, 2, \ldots, n\}$ that is weakly increasing along rows, strictly increasing up columns, and satisfies $|T^{-1}(\{i\})| = \alpha_i$ for $1 \le i \le \ell(\alpha)$. (Intuitively, this means the tableau has α_i copies of i for each i.) The *standardization* of $T \in \mathrm{SSYT}(\lambda, (\alpha_1, \ldots, \alpha_k))$ is obtained by replacing the α_j copies of j in T, in the order they appear in $\mathrm{rw}(T)$, by $m, m+1, \ldots, m+\alpha_j - 1$ where $m = 1 + \sum_{i=1}^{j-1} \alpha_i$.

The classical Kostka matrix, which is square of order $p(n) \times p(n)$, is obtained from our matrix K_n by erasing columns indexed by compositions that are not partitions.

Example 2. When $n = 4$, our Kostka matrix is

$$K_4 = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{array} \begin{array}{c} \begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{array} \\ \left(\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)_{5\times 8} \end{array}.$$

The classical square Kostka matrix for $n = 4$ is

$$K_4^c = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 31 & 22 & 211 & 1111 \end{array} \\ \left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)_{5\times 5} \end{array}.$$

We use a superscript c to denote the classical (square) versions of the Kostka matrix and the inverse Kostka matrix.

4 The Inverse Kostka Matrix

In this section, we define a rectangular matrix K'_n of order $c(n) \times p(n)$ that turns out to be a *right inverse* of K_n. Our discussion here is based on the treatment in [5], but with a few modifications. Given $\alpha \models n$ and $\lambda \vdash n$, we define $K'_n(\alpha, \lambda)$ to be the sum[1] of the signs of the special rim-hook tableaux of shape λ that have nonzero rim-hook lengths *in order from bottom to top* given by $\alpha_1, \ldots, \alpha_s$. Recall that a rim-hook tableau is called *special* iff every rim-hook begins in the leftmost column. Also, the sign of a rim-hook spanning r rows is $(-1)^{r-1}$, and the sign of a rim-hook tableau is the product of the signs of the rim-hooks in it.

Example 3. For $\lambda = (5, 5, 3, 2)$ and $\alpha = (4, 3, 8)$, we have $K'_{15}(\alpha, \lambda) = +1$. The relevant special rim-hook tableau is shown in Figure 1.

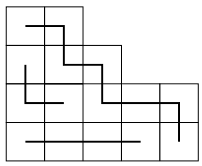

Figure 1: A special rim-hook tableau.

Example 4. When $n = 4$, our inverse Kostka matrix is

$$K'_4 = \begin{array}{cc} & \begin{array}{ccccc} 4 & 31 & 22 & 211 & 1111 \end{array} \\ \begin{array}{c} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} & \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{8\times 5} \end{array}. \tag{3}$$

In contrast, the classical inverse Kostka matrix used in [5] is a square matrix of order $p(n) \times p(n)$, where the μ, λ-entry is the sum $\sum_{\alpha:\, \mathrm{sort}(\alpha)=\mu} K'_n(\alpha, \lambda)$ over all compositions α that are rearrangements of the partition μ. For $n = 4$, this leads to the matrix

$$K'^c_4 = \begin{array}{cc} & \begin{array}{ccccc} 4 & 31 & 22 & 211 & 1111 \end{array} \\ \begin{array}{c} 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{array} & \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{5\times 5} \end{array}.$$

[1]We shall see below that the sum contains at most one term.

In [5], Eğecioğlu and Remmel proved that $K_n^c K_n'^c = I_{p(n)}$. Proposition 5 follows immediately from their proof; we sketch the argument for completeness.

Proposition 5. *For all $n \geq 1$, $K_n K_n' = I_{p(n)}$.*

Proof. We must prove that, for all $\lambda, \mu \vdash n$,

$$\sum_{\alpha \models n} K_n(\lambda, \alpha) K_n'(\alpha, \mu) = \chi(\lambda = \mu).$$

But this is *exactly* what Eğecioğlu and Remmel prove in [5], using a sign-reversing involution on pairs (T, S) where $T \in \mathrm{SSYT}(\lambda, \alpha)$ and S is a special rim-hook tableau of shape μ and content α. Their proof requires an extra step that converts the previous identity to the companion identity

$$\sum_{\nu \vdash n} K_n^c(\lambda, \nu) K_n'^c(\nu, \mu) = \chi(\lambda = \mu).$$

This step uses the Bender-Knuth bijections on semistandard tableaux [3] that prove $K_n(\lambda, \alpha) = K_n(\lambda, \beta)$ whenever β is a rearrangement of α. □

The next proposition reveals another simplification that arises from using the rectangular version of the inverse Kostka matrix.

Proposition 6. *Every entry of K_n' is $+1$, 0, or -1. Moreover, these entries can be computed by the recursion* (4) *described below.*

Proof. The result follows because there is at most one way to fill a partition diagram $\mathrm{dg}(\lambda)$ using the *ordered* sequence of special rim-hook lengths $\alpha = (\alpha_1, \ldots, \alpha_s)$. More explicitly, we see by drawing a picture that the last special rim-hook (of length α_s) can be placed in λ iff $\lambda_k + (\ell(\lambda) - k) = \alpha_s$ for some (necessarily unique) $k \leq \ell(\lambda)$. If this condition fails, then $K_n'(\alpha, \lambda) = 0$. Otherwise, stripping off this special rim-hook shows that

$$K_n'(\alpha, \lambda) = (-1)^{\ell(\lambda)-k} K_{n-\alpha_s}'(\alpha^*, \lambda^*), \tag{4}$$

where $\alpha^* = (\alpha_1, \ldots, \alpha_{s-1})$ and $\lambda^* = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}-1, \ldots, \lambda_{\ell(\lambda)}-1)$ with zero parts deleted; the initial condition is $K_0'(0, 0) = 1$. The theorem now follows by induction. □

5 Inclusion-Exclusion Adjustment

To connect the preceding material to equation (2), we introduce two new matrices, M_n and A_n. First, M_n is the $p(n) \times c(n)$ matrix defined by

$$M_n(\lambda, \alpha) = |\{T \in \mathrm{SYT}(\lambda) : \mathrm{Des}'(T) = \alpha\}| \qquad (\lambda \vdash n, \alpha \models n).$$

Second, A_n is the $c(n) \times c(n)$ matrix defined by

$$A_n(\alpha, \beta) = \chi(\beta \text{ is finer than } \alpha) \qquad (\alpha, \beta \models n).$$

Recall that $\beta = (\beta_1, \ldots, \beta_t)$ is *finer* than $\alpha = (\alpha_1, \ldots, \alpha_s)$ iff there exist indices $0 = i_0 < i_1 < i_2 < \cdots < i_s = t$ such that $\alpha_j = \sum_{i:\, i_{j-1} < i \leq i_j} \beta_i$ for $1 \leq j \leq s$; in this case, we also say α is *coarser* than β.

Example 7. When $n = 4$,

$$
M_4 = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}_{5\times 8}
\end{array}
$$

and

$$
A_4 = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}_{8\times 8}
\end{array}.
$$

Remark 8. We have $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and it is routine to check that $A_n = A_2^{\otimes(n-1)}$ for $n \geq 2$. That is, A_n is the iterated tensor product (or Kronecker product) of $n-1$ copies of A_2.

Lemma 9. *For all $n \geq 1$, $M_n A_n = K_n$.*

Proof. We must show

$$\sum_{\alpha \models n} M(\lambda, \alpha) A(\alpha, \beta) = K(\lambda, \beta)$$

for each fixed $\lambda \vdash n, \beta \models n$. The summation here is the size of the set

$$X = \bigcup_{\alpha} \{T \in \mathrm{SYT}(\lambda) : \mathrm{Des}'(T) = \alpha\}$$

where the union extends over all α coarser than β. On the other side, $K(\lambda, \beta)$ is the size of the set $Y = \mathrm{SSYT}(\lambda, \beta)$. We define a bijection $F : Y \to X$ by sending a semistandard tableau T of shape λ and content β to its standardization (in the usual sense); one checks that the descent composition of $\mathrm{std}(T)$ is always some composition α coarser than β. The inverse bijection $F^{-1} : X \to Y$ is obtained by "unstandardizing" a standard tableau S of shape λ to get a semistandard tableau of the given content β. Since β is fixed and known, this can be done in exactly one way provided that the descent composition of S is coarser than β. $\square$

Example 10. Suppose $\lambda = (4, 3, 2)$ and $\beta = (2, 1, 2, 1, 1, 2)$. Then

$$
F\left(\begin{array}{|c|c|cc} \hline 3 & 6 & \multicolumn{2}{|c}{} \\ \hline 2 & 4 & \multicolumn{1}{c|}{6} & \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array}\right) = \begin{array}{|c|c|cc} \hline 4 & 8 & \multicolumn{2}{|c}{} \\ \hline 3 & 6 & \multicolumn{1}{c|}{9} & \\ \hline 1 & 2 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{7} \\ \hline \end{array}\,, \qquad
F^{-1}\left(\begin{array}{|c|c|cc} \hline 8 & 9 & \multicolumn{2}{|c}{} \\ \hline 4 & 6 & \multicolumn{1}{c|}{7} & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array}\right) = \begin{array}{|c|c|cc} \hline 6 & 6 & \multicolumn{2}{|c}{} \\ \hline 3 & 4 & \multicolumn{1}{c|}{5} & \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array}\,.
$$

6 Schur versus Quasisymmetric Expansions

Define $K_n^* = A_n K_n'$. This is a $c(n) \times p(n)$ matrix for which

$$K_n^*(\alpha, \lambda) = \sum_{\beta \text{ finer than } \alpha} K_n'(\beta, \lambda).$$

We can now prove a matrix version of (2), which leads directly to the desired formula (1).

Theorem 11. *Suppose F is a field, and we have a symmetric function*

$$f = \sum_{\lambda \vdash n} x_\lambda s_\lambda = \sum_{\alpha \models n} y_\alpha Q_\alpha \qquad (x_\lambda, y_\alpha \in F).$$

The row vectors $\mathbf{x} = (x_\lambda : \lambda \vdash n)$ and $\mathbf{y} = (y_\alpha : \alpha \models n)$ satisfy

$$\mathbf{x} M_n = \mathbf{y} \quad \textit{and} \quad \mathbf{x} = \mathbf{y} K_n^*.$$

So, $x_\lambda = \sum_{\alpha \models n} y_\alpha K_n^*(\alpha, \lambda)$ for all $\lambda \vdash n$.

Proof. By definition of M_n, we can rewrite (2) as $s_\lambda = \sum_{\alpha \models n} M_n(\lambda, \alpha) Q_\alpha$. So,

$$f \;=\; \sum_{\lambda \vdash n} x_\lambda s_\lambda = \sum_{\lambda \vdash n} x_\lambda \sum_{\alpha \models n} M_n(\lambda, \alpha) Q_\alpha = \sum_{\alpha \models n} Q_\alpha \left(\sum_{\lambda \vdash n} x_\lambda M_n(\lambda, \alpha) \right).$$

But we also have $f = \sum_{\alpha \models n} Q_\alpha y_\alpha$. Since the Q_α's are linearly independent, equating coefficients gives $\mathbf{x} M_n = \mathbf{y}$. Multiplying both sides on the right by K_n^* gives

$$\mathbf{y} K_n^* = \mathbf{x} M_n A_n K_n' = \mathbf{x} K_n K_n' = \mathbf{x}.$$

□

Example 12. The matrices K_n^* are remarkably sparse. When $n = 4$, we find

$$K_4^* = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 31 & 22 & 211 & 1111 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{8 \times 5} \end{array}.$$

This means that $x_{22} = y_{22} - y_{13}$, whereas $x_\lambda = y_\lambda$ for all other partitions λ of 4. In general, we will see below that $x_\lambda = y_\lambda$ whenever λ is a hook partition. When $n = 5$, the equations for non-hook partitions are $x_{32} = y_{32} - y_{14}$ and $x_{221} = y_{221} - y_{131}$. For $n = 6$, the equations are:

$$\begin{aligned} x_{42} &= y_{42} - y_{15} \\ x_{33} &= y_{33} - y_{24} \\ x_{321} &= y_{321} - y_{141} \\ x_{2211} &= y_{2211} - y_{1311} \\ x_{222} &= y_{222} - y_{213} - y_{132} + y_{114}. \end{aligned}$$

Remark 13. The matrix M_n can be viewed as the matrix (relative to the bases $\{s_\lambda\}$ and $\{Q_\alpha\}$) of the linear map ι which is the inclusion of Sym_n into QSym_n. Similarly, the matrix K_n^* can be viewed as the matrix of a certain linear projection $\pi : \mathrm{QSym}_n \to \mathrm{Sym}_n$ satisfying $\pi \circ \iota = \mathrm{id}_{\mathrm{Sym}_n}$. In general, we could start with any $f \in \mathrm{QSym}_n$ and calculate the coordinates x_λ of $\pi(f)$ by multiplying by K_n^* as above. In our applications, f is actually known to be in Sym_n, so $\pi(f) = f$. The projection π is not unique, but the combinatorial proof of $K_n K_n' = I$ suggests that π is "natural" in some sense. Observe that the $c(n) \times c(n)$ matrix $K_n' K_n$ is *not* the identity matrix of order $c(n)$. Rather, the entries of this matrix specify the relations between the y_α's that must be satisfied in order for $f = \sum_\alpha y_\alpha Q_\alpha$ to be symmetric (not merely quasisymmetric). It is not hard to show that symmetry holds iff $y_\alpha = y_\beta$ whenever β is a rearrangement of α; but the matrix $K_n' K_n$ encodes these conditions in a rather subtle way, as one discovers by looking at examples.

7 Combinatorial Meaning of K_n^*

In order to give a combinatorial interpretation to the entries of K_n^*, we introduce the following definitions. Let $\alpha \models n,\ \lambda \vdash n$. We say that α is *realizable* (for λ) if there exists a (necessarily unique) special rim-hook tableau S of shape λ and content α. We denote the special rim-hooks of S by $r_1, r_2, \ldots, r_{\ell(\alpha)}$. If, furthermore, each rim-hook of S contains exactly one cell in the first column of the Ferrers diagram of λ, then we say that (α, λ) (or, equivalently, S) is *flat*. The following lemma is presented without proof.

Lemma 14. *Let S be a special rim-hook tableau of shape λ and content α. Suppose r_{i-1} and r_i are consecutive rim-hooks in the special rim-hook tableau S. Then either*

1. *the last cell of r_i is immediately above a cell of r_{i-1}, or*

2. *the last cell of r_{i-1} is immediately to the left of a cell of r_i.*

Theorem 15. *Let $\alpha \models n,\ \lambda \vdash n$. If (α, λ) is flat, then $K_n^*(\alpha, \lambda) = K_n'(\alpha, \lambda) = \pm 1$. Otherwise, $K_n^*(\alpha, \lambda) = 0$. In particular, $K_n^*(\alpha, \lambda) = \chi(\alpha = \lambda)$ when λ is a hook.*

Proof. First consider the case where (α, λ) is flat. Then $\ell(\alpha) = \ell(\lambda)$. Any $\beta \models n$ that is strictly finer than α must satisfy $\ell(\beta) > \ell(\alpha)$. However, the pigeonhole principle then implies that there are no special rim-hook tableaux of shape λ and content β. The result follows in this case by Proposition 6 and the definition of K_n^*.

We may now assume that (α, λ) is not flat. Let

$$C = \{\beta \models n : \beta \text{ is finer than } \alpha \text{ and is realizable for } \lambda\}.$$

In the case that (α, λ) is not realizable, it is possible that $C = \emptyset$, in which case $K_n^*(\alpha, \lambda) = 0$ and we are done. So assume $C \neq \emptyset$. We will define a sign-reversing involution ψ on C.

Fix $\beta \in C$. By definition, β is realized by a special rim-hook tableau S with rim-hooks $r_1, r_2, \ldots, r_{\ell(\beta)}$. Since β is finer than α, there exist indices $0 = i_0 < i_1 < i_2 < \cdots < i_s = t$ for which $\alpha_j = \sum_{i:\ i_{j-1} < i \le i_j} \beta_i$ for $1 \le j \le s$. We say the parts $\beta_{i_{j-1}+1}, \beta_{i_{j-1}+2}, \ldots, \beta_{i_j}$ of β *contribute* to the part α_j of α.

Each cell $(m, 1)$ in the first column of S is part of some special rim-hook r_i (not necessarily starting in the m-th row) of length β_i. Label the m-th row of S by the index of the part of α to which β_i contributes. A label k will repeat whenever $i_k - i_{k-1} > 1$ or when r_{i_k} travels vertically in the first column. Let j be the maximum repeated label. Visually, j is the maximum index for which $r_{i_{j-1}+1}, r_{i_{j-1}+2}, \ldots, r_{i_j}$ collectively occupy at least two cells in the first column. Note that a repeated label must exist: otherwise $\beta = \alpha$ and hence (α, λ) is flat, contradicting our assumption. That the map ψ defined below is an involution follows from the fact that ψ does not change the labeling of the rows.

We write $C = C_1 \cup C_2 \cup C_3 \cup C_4$ where the C_i are defined below. (This is not a set partition of C: while the C_i are pairwise disjoint, we do allow $C_i = \emptyset$.)

1. $C_1 = \{\beta \in C : \beta_{i_j} = 1\}$.
2. $C_2 = \{\beta \in C : r_{i_j}$ contains at least two cells in the first column$\}$.
3. $C_3 = \{\beta \in C : \beta_{i_j} \geq 2$, r_{i_j} contains only one cell in the first column and r_{i_j} ends as in Lemma 14.1$\}$.
4. $C_4 = \{\beta \in C : \beta_{i_j} \geq 2$, r_{i_j} contains only one cell in the first column and $r_{i_j - 1}$ ends as in Lemma 14.2$\}$.

The r_i corresponding to the relevant β_i are illustrated in Figure 2. For C_1 and C_2, we show a portion of the first column of the Ferrers diagram, whereas for C_3 and C_4, we show how the rim-hooks in question interact at the end of the shorter one. Dotted lines indicate possible, unspecified extensions of the rim-hooks.

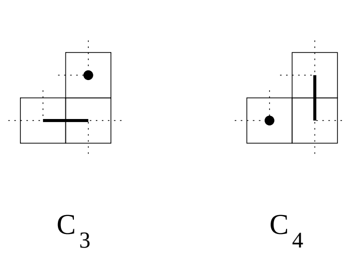

Figure 2: Possible interactions between consecutive rim-hooks.

We now define ψ. It follows from the definitions below that $\psi(C_1) = C_2$, $\psi(C_2) = C_1$, $\psi(C_3) = C_4$ and $\psi(C_4) = C_3$. For brevity, write γ for $\psi(\beta)$. For all cases, we define $\gamma_k = \beta_k$ for $k < i_j - 1$.

$$\beta \in C_1 : \quad \gamma_k = \begin{cases} \beta_k + 1, & k = i_j - 1, \\ \beta_{k+1}, & i_j \leq k < \ell(\beta). \end{cases}$$

$$\beta \in C_2 : \quad \gamma_k = \begin{cases} \beta_k - 1, & k = i_j, \\ 1, & k = i_j + 1, \\ \beta_{k-1}, & i_j + 1 < k \leq \ell(\beta) + 1. \end{cases}$$

$$\beta \in C_3 \cup C_4 : \quad \gamma_k = \begin{cases} \beta_{k-1} + 1, & k = i_j, \\ \beta_{k+1} - 1, & k = i_j - 1, \\ \beta_k, & i_j < k \leq \ell(\beta). \end{cases}$$

This is sign-reversing since exactly one vertical bond is either created or broken. The sum of the length(s) of the special rim-hook(s) starting in these two rows is left unchanged, so our new composition is still finer than α. Finally, our new composition is manifestly realizable. This completes the proof. $\square$

Example 16. Let $\lambda = (4, 4, 4, 4, 3, 2, 1, 1)$ and $\beta = (1, 5, 5, 8, 2, 1, 1)$.

- If $\alpha = (1, 5, 5, 8, 2, 2)$ then $\beta \in C_1$ and $\psi(\beta) = (1, 5, 5, 8, 2, 2)$.
- If $\alpha = (6, 5, 8, 2, 1, 1)$, then $\beta \in C_2$ and $\psi(\beta) = (1, 4, 1, 5, 8, 2, 1, 1)$.
- If $\alpha = (1, 18, 2, 1, 1)$ then $\beta \in C_4$ and $\psi(\beta) = (1, 5, 7, 6, 2, 1, 1)$.

The third example is illustrated in Figure 3. The row labels indicate the part of α to which each rim-hook contributes.

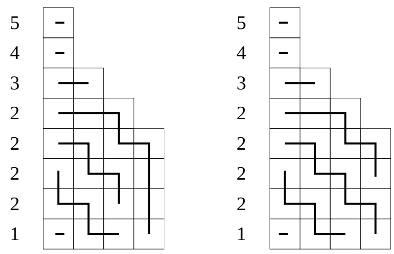

Figure 3: Example of the involution ψ.

Remark 17. Given any flat (α, λ) with $\lambda \vdash n$, let $\tilde{\lambda} \vdash \tilde{n} < n$ be the partition obtained by decrementing each part of λ by 1 and then removing any resulting parts of size 0. Define $\tilde{\alpha}$ analogously. It is immediate that $(\tilde{\alpha}, \tilde{\lambda})$ is realizable by a special rim hook tableau of the same sign as the one realizing (α, λ). Equivalently, $K^*_n(\alpha, \lambda) = K'_{\tilde{n}}(\tilde{\alpha}, \tilde{\lambda})$ whenever (α, λ) is flat. If we wish to extend this association to $K'^c_{\tilde{n}}$, we must fix some $\mu \vdash n$. Then

$$\sum_{\alpha:\, \mathrm{sort}(\alpha)=\mu} K^*_n(\alpha, \lambda) = K'^c_{\tilde{n}}(\tilde{\mu}, \tilde{\lambda}). \tag{5}$$

8 Enumerative aspects of K^*_n

Let λ be an integer partition with $\ell(\lambda)$ parts; set $\lambda_k = 0$ for all $k > \ell(\lambda)$. An *inner corner* of the Ferrers diagram $\mathrm{dg}(\lambda)$ is a cell (i, λ_i) for which $\lambda_{i+1} < \lambda_i$ (i.e., a cell whose removal leaves a valid Ferrers diagram). An *outer corner* is a cell $(i, \lambda_i + 1)$ for which $\lambda_i < \lambda_{i-1}$. Write

$$\underline{\mathrm{srht}}(\lambda) = \sum_{\alpha \models n} |K^*_n(\alpha, \lambda)| \quad \text{(resp.} \quad \mathrm{srht}(\lambda) = \sum_{\alpha \models n} |K'_n(\alpha, \lambda)|) \tag{6}$$

for the number of flat (resp. arbitrary) special rim-hook tableaux of shape λ. Using Remark 17, we see that for any partition λ,

$$\underline{\mathrm{srht}}(\lambda) = \mathrm{srht}(\tilde{\lambda}).$$

Lemma 18. *Let $\lambda \vdash n$ and suppose (c, λ_c), $c < \lambda_c$, is an inner corner of* $\mathrm{dg}(\lambda)$. *Write λ^- for the partition of $n-1$ obtained by removing this cell (c, λ_c). Then* $\underline{\mathrm{srht}}(\lambda) = \underline{\mathrm{srht}}(\lambda^-)$ *and* $\mathrm{srht}(\lambda) = \mathrm{srht}(\lambda^-)$.

Proof. First, we prove $\underline{\mathrm{srht}}(\lambda) = \underline{\mathrm{srht}}(\lambda^-)$ by constructing a bijection between sets of realizable pairs $\{(\alpha, \lambda)\}$ and $\{(\beta, \lambda^-)\}$. Begin with a pair (α, λ). Write δ for the difference $\lambda_c - c$. Note that two cells $(i, i+\delta)$ and $(j, j+\delta)$, $i \neq j$ must be part of different special rim-hooks. However, since (c, λ_c) is an inner corner and $c < \lambda_c$, there are only $c-1$ cells along the diagonal $(i, i+\delta+1)$ into which these c rim-hooks can extend. It follows that there exists a maximum value d for which some rim-hook ends in the cell $(d, d+\delta)$.

To construct the pair (β, λ^-), first shorten the special rim-hook passing through $(d, d+\delta)$ by one unit. Then, for $i > d$ reroute each rim-hook as illustrated in Figure 4.

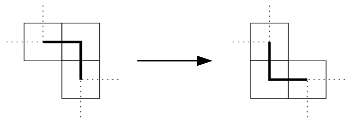

Figure 4: Rerouting hooks to fit inside $\mathrm{dg}(\lambda^-)$.

To invert, look for the maximum d for which a rim-hook ends in cell $(d, d+\delta-1)$. Such a d must exist by the above pigeonhole argument. The rerouting of Figure 4 is then performed in reverse. A full example is illustrated in Figure 5.

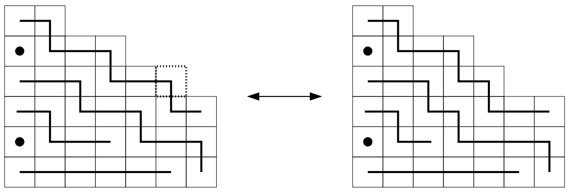

Figure 5: Example of matching between (α, λ) and (β, λ^-) for two flat pairs.

No rim-hook has been modified within the first column, so (β, λ^-) will be flat if and only if (α, λ) is flat. $\square$

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ be a partition of n. Set $\lambda^i = (\lambda_{k-i+1} \geq \lambda_{k-i+2} \geq \cdots \geq \lambda_k)$ to be the partition formed by the *last* i parts of λ. The *Durfee square size*, $\mathrm{DF}(\lambda)$, of a partition λ is the largest m for which $\lambda_m \geq m$.

Theorem 19. *For all partitions λ, the number of flat special rim-hook tableaux of shape λ is* $\prod_{i=1}^{k} \mathrm{DF}(\lambda^i)$.

Proof. We argue by induction on the number of parts of λ. For our induction hypothesis, we assume the number of flat special rim-hook tableaux of shape λ^{k-1} is given by $\prod_{i=1}^{k-1} \mathrm{DF}(\lambda^i)$. Consider any such flat pair $((\alpha_1, \ldots, \alpha_{k-1}), \lambda^{k-1})$ embedded in the top $k-1$ rows of λ. (All row indices will be with respect to the shape λ.)

By Lemma 18, we can assume $\lambda_1 = \lambda_2 = \cdots = \lambda_{\mathrm{DF}(\lambda)} = \mathrm{DF}(\lambda)$. Insert a special rim-hook r_0 of length β_0, $1 \leq \beta_0 \leq \mathrm{DF}(\lambda)$ into the first row of $\mathrm{dg}(\lambda)$. (Since the rim-hook r_0 starts in the bottom row, it is necessarily horizontal.) Let $i_1, \ldots, i_{\mathrm{DF}(\lambda)-\beta_0}$ index those rim-hooks containing a cell of the form $(c, c+\beta_0-1)$ for $c>1$. Increase each α_{i_j} by one. This can be visualized by splicing in a vertical segment to each rim-hook r_{i_j} just below the corresponding cell $(c, c+\beta_0-1)$. The new composition created is flat for λ. See Figure 6 for an example with $\beta_0=2$ and $(\alpha_1,\ldots,\alpha_{k-1})=(4,3,7,6,2,11,1,8)$.

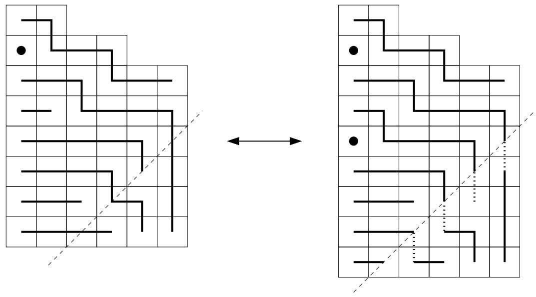

Figure 6: Example of adding a new rim-hook along the bottom.

To invert this map, remove β_0 and excise the length-one vertical segment abutting each cell of the form $(c, c+\beta_0-1)$ for $c>1$. (Note that by a similar pigeonhole argument to that used previously, no rim-hook can end in, or exit to the right from, any of these cells $(c, c+\beta_0-1)$ for $c>1$.) ☐

Corollary 20. *Fix k with $\sqrt{n}<k\le n$. The maximum of $\underline{\mathrm{srht}}(\lambda)$ over all partitions $\lambda\vdash n$ with k parts is attained by $\lambda=\nu$ where $\nu^t=(k,k,\ldots,k,n \bmod k)$. Writing $d=\lfloor n/k\rfloor$ and $a=\min(k-(n \bmod k), k-d)$,*

$$\underline{\mathrm{srht}}(\nu)=d!d^a(d+1)^{k-d-a}\le(d+1)^k. \tag{7}$$

If instead $k\le\sqrt{n}$, then the maximum of $\underline{\mathrm{srht}}(\nu)$ is $k!$, and this maximum is obtained for any ν such that $\nu_1^t=\cdots=\nu_k^t=k$.

Proof. Assume $\sqrt{n}<k\le n$, and let λ be any partition of n with $\ell(\lambda)=k$. We show that if λ^t is *not* of the form $(k,k,\ldots,k,n \bmod k)$, then we can find a partition $\mu\vdash n$ dominated by λ for which $\ell(\mu)=k$ and $\underline{\mathrm{srht}}(\mu)\ge\underline{\mathrm{srht}}(\lambda)$.

Indeed, if λ^t is not of the form $(k,k,\ldots,k,n \bmod k)$, then it must have at least one inner corner (i,λ_i) and at least one outer corner (j,λ_j+1) with $k\ge j>i$ and $\lambda_j<\lambda_i-1$. Let λ^- denote the partition of $n-1$ obtained by removing the cell in position (i,λ_i) from $\mathrm{dg}(\lambda)$ and let μ denote the partition of n obtained by adding the outer corner (j,λ_j+1) to $\mathrm{dg}(\lambda^-)$.

If $j<\lambda_j+1$ then necessarily $i<\lambda_i$. It then follows from Lemma 18 that $\underline{\mathrm{srht}}(\mu)=\underline{\mathrm{srht}}(\lambda)$. So we assume that $j\ge\lambda_j+1$. By Theorem 19, $\underline{\mathrm{srht}}(\mu)=\frac{\lambda_j+1}{\lambda_j}\underline{\mathrm{srht}}(\lambda^-)$. If $i\ge\lambda_i$, we similarly have $\underline{\mathrm{srht}}(\lambda)=\frac{\lambda_i}{\lambda_i-1}\underline{\mathrm{srht}}(\lambda^-)$. If $i<\lambda_i$, then $\underline{\mathrm{srht}}(\lambda)=\underline{\mathrm{srht}}(\lambda^-)$. In either case, since $\lambda_j<\lambda_i$, we find that $\underline{\mathrm{srht}}(\mu)\ge\underline{\mathrm{srht}}(\lambda)$.

Formula (7) is a direct result of Theorem 19; its derivation is left to the reader. The proof for the case $k\le\sqrt{n}$ is also routine and will be omitted. ☐

Theorem 21. *Define a generating function $A(q)=\sum_n\sum_{\lambda\vdash n}\underline{\mathrm{srht}}(\lambda)q^n$. Then*

$$A(q)=\sum_{k=0}^{\infty}k!q^{k^2}\prod_{j=1}^{k}\frac{1}{1-jq^j}\prod_{j=1}^{k}\frac{1}{1-q^j}=\sum_{k=0}^{\infty}q^{\binom{k}{2}}\prod_{j=1}^{k}\frac{jq^j}{1-jq^j}\prod_{j=1}^{k}\frac{1}{1-q^j}. \tag{8}$$

Proof. We will derive the rightmost expression in (8) for $A(q)$. Summing over k amounts to partitioning λ according to its Durfee square size k. Fix some $k \geq 0$ and consider the staircase of lightly-shaded cells as in Figure 7. These $\binom{k}{2}$ cells in the first k columns and below the diagonal $y = x$ must be present for any partition of Durfee square size k; they are accounted for by the $q^{\binom{k}{2}}$ in (8).

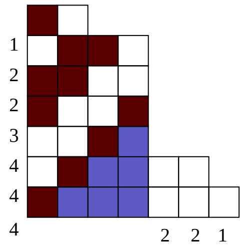

Figure 7: Decomposition of $\lambda = (7, 6, 4, 4, 4, 4, 2)$.

The heights of the columns of cells to the right of the k-th column must be weakly decreasing and the first column must have height at most k. The multiplicities for each height can be chosen freely and independently. The term $(q^j)^{a_j}$ in the expansion $(1 - q^j)^{-1} = 1 + q^j + (q^j)^2 + \cdots$ corresponds to choosing a_j columns of height j. The coefficient of $q^{a_k + a_{k-1} + \cdots + a_1}$ in the expansion of this last product is 1 as desired since these cells do not affect $\underline{\mathrm{srht}}(\lambda)$.

The cells weakly above the diagonal $y = x$ will be grouped according to the diagonal $y = x + c$ they lie upon. As above, these multiplicities b_j can be chosen independently. However, here we must have each $b_j \geq 1$. This restriction accounts for the q^j in the middle product of (8). The coefficient j of each q^j accounts for the fact that such a diagonal of cells corresponds to one of the $\lambda^{i'}$ having Durfee square size j. It follows that the coefficient of $q^{b_k + b_{k-1} + \cdots + b_1}$ will be $k^{b_k}(k-1)^{b_{k-1}} \cdots 1^{b_1} = \mathrm{DF}(\lambda)$. A full example for $k = 4$, $a_4 = a_3 = 0$, $a_2 = 2$, $a_1 = 1$, $b_4 = 3$, $b_3 = 1$, $b_2 = 2$, $b_1 = 1$ is shown in Figure 7. The darkly-shaded cells correspond to the restrictions $b_j \geq 1$. $\square$

Remark 22. We can add a power of t to keep track of $\ell(\lambda)$, by replacing each occurrence of jq^j on the far right side of (8) by jtq^j.

Remark 23. Table 1 contains the first few coefficients of $A(q)$ and the corresponding coefficients in the generating function for arbitrary special rim-hook tableaux. If we view (8) as a function of the complex variable q, one can use the Weierstrass M-test and the ratio test to show that the series converges uniformly on the open disk $\{q \in \mathbb{C} : |q| < 3^{-1/3}\}$. So $A(q)$ is an analytic function on this domain, but $A(q)$ has a singularity at $q = 3^{-1/3}$. It follows that $\sum_{\lambda \vdash n} \underline{\mathrm{srht}}(\lambda)$ is in $O(3^{n/3})$.

Table 1: Numbers of flat and arbitrary special rim-hook tableaux.

n	1	2	3	4	5	6	7	8	9	10	11
$\underline{\mathrm{srht}}(n)$	1	2	3	6	9	18	27	50	79	138	215
$\mathrm{srht}(n)$	1	3	7	17	37	85	181	399	841	1805	3757

9 Application to Schur Expansion of Macdonald Polynomials

Let μ be a fixed integer partition of n. In [7], Haglund defined two permutation statistics $\mathrm{inv}_\mu, \mathrm{maj}_\mu : S_n \to \mathbb{N}$, which are now known [8] to give the "Macdonald polynomial Hilbert series"

$$\langle \tilde{H}_\mu, h_{(1^n)} \rangle = \sum_{w \in S_n} q^{\mathrm{inv}_\mu(w)} t^{\mathrm{maj}_\mu(w)}.$$

Here, we are identifying Haglund's "standard fillings" (bijections from $\mathrm{dg}(\mu)$ to $\{1, 2, \ldots, n\}$) with their associated reading words $w \in S_n$. For example, if $n = 7$ and $\mu = (3, 2, 1, 1)$, the standard filling

$$\ytableaushort{3,7,54,162}$$

is identified with its reading word $w = 3754162$; and $\mathrm{inv}_\mu(w) = 2$, $\mathrm{maj}_\mu(w) = 5$.

Haglund's combinatorial formula for $\tilde{H}_\mu$ actually gives the expansion of this symmetric polynomial in the fundamental quasisymmetric basis Q_α. Given $w \in S_n$, let $\mathrm{IDes}(w) = \mathrm{Des}(w^{-1})$ be the set of $i < n$ such that $i + 1$ appears to the left of i in $w_1 w_2 \cdots w_n$. Writing $\mathrm{IDes}(w) = \{i_1 < i_2 < \cdots < i_k\}$, we set $\mathrm{IDes}'(w) = (i_1, i_2 - i_1, \ldots, n - i_k) \models n$ (cf. the earlier discussion of $\mathrm{Des}(T)$ and $\mathrm{Des}'(T)$ for standard tableaux). It is known, via standardization bijections, that

$$\tilde{H}_\mu = \sum_{w \in S_n} q^{\mathrm{inv}_\mu(w)} t^{\mathrm{maj}_\mu(w)} Q_{\mathrm{IDes}'(w)}. \tag{9}$$

In particular, [8, Theorem 3.1] proves that the quasisymmetric function on the righthand side of (9) is symmetric. Therefore, the results in §6 and §7 specialize as follows.

Theorem 24. *For all $\lambda, \mu \vdash n$,*

$$\langle \tilde{H}_\mu, s_\lambda \rangle = \sum_{\alpha \models n} K_n^*(\alpha, \lambda) \left(\sum_{w \in S_n : \mathrm{IDes}'(w) = \alpha} q^{\mathrm{inv}_\mu(w)} t^{\mathrm{maj}_\mu(w)} \right). \tag{10}$$

So, if λ is a hook, then the coefficient of s_λ in $\tilde{H}_\mu$ is the sum of $q^{\mathrm{inv}_\mu(w)} t^{\mathrm{maj}_\mu(w)}$ over those $w \in S_n$ such that $\mathrm{IDes}'(w) = \lambda$.

Example 25. Let $\mu = (2, 1, 1)$ and $\lambda = (2, 2)$. The compositions α of 4 for which $(\alpha, (2, 2))$ is flat are $(2, 2)$ and $(1, 3)$. We have

$$K^*((2, 2), (2, 2)) = +1 \text{ and } K^*((1, 3), (2, 2)) = -1.$$

Listing the $w \in S_n$ for which $\mathrm{IDes}'(w)$ is $(2, 2)$ or $(1, 3)$ and computing the statistics $\mathrm{inv}_{(2,1,1)}$ and $\mathrm{maj}_{(2,1,1)}$ produces the fillings shown here.

α	$(2,2)$					$(1,3)$		
sign	$+1$					-1		
Filling	$\ytableaushort{3,4,12}$	$\ytableaushort{3,1,42}$	$\ytableaushort{3,1,24}$	$\ytableaushort{1,3,24}$	$\ytableaushort{1,3,42}$	$\ytableaushort{2,3,41}$	$\ytableaushort{2,3,14}$	$\ytableaushort{2,1,34}$
$\mathrm{inv}_{(2,1,1)}$	0	1	0	0	1	1	0	0
$\mathrm{maj}_{(2,1,1)}$	2	1	1	2	0	0	2	1

Summing terms yields $\langle \tilde{H}_{(2,1,1)}, s_{(2,2)} \rangle = t^2 + tq$.

To prove Schur positivity of $\tilde{H}_\mu$ by our technique, one must define involutions to cancel all negative terms on the right side of (10). This seems to be a difficult problem, even when the right side is the difference of just two terms.

Fundamental quasisymmetric expansions are also known for LLT polynomials and (conjecturally) for symmetric functions of the form $\nabla(s_\lambda)$ [10]. Thus, Theorem 11 also applies to give (signed) combinatorial formulas for the Schur expansions of these symmetric functions.

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