Bivariate Generating Functions for Involutions Restricted by 3412*

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Abstract

We study bivariate generating functions for the number of involutions in S_n subject to two restrictions. One restriction is that the involution avoid 3412 or contain 3412 exactly once. The other restriction is that the involution avoid another pattern π or contain π exactly once. In many cases we express these generating functions in terms of Chebyshev polynomials of the second kind.

Keywords: Restricted permutation; restricted involution; pattern-avoiding permutation; forbidden subsequence; Chebyshev polynomial

1 Introduction and Notation

Let S_n denote the set of permutations of $\{1, \ldots, n\}$, written in one-line notation, and suppose $\pi \in S_n$. We write $|\pi|$ to denote the length of π, and for all i, $1 \leq i \leq n$, we write $\pi(i)$ to denote the ith element of π. We say π is an *involution* whenever $\pi(\pi(i)) = i$ for all i, $1 \leq i \leq n$, and we write I_n to denote the set of involutions in S_n. Now suppose $\pi \in S_n$ and $\sigma \in S_k$. We say a subsequence of π has *type* σ whenever it has all of the same pairwise comparisons as σ. For example, the subsequence 2869 of the permutation 214538769 has type 1324. We say π *avoids* σ whenever π contains no subsequence of type σ. In this context σ is sometimes called a *pattern* or *forbidden subsequence* and π is sometimes called

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a *restricted permutation* or *pattern-avoiding permutation*. In this paper we will be interested in involutions which avoid several patterns, so for any set R of permutations we write $I_n(R)$ to denote the set of involutions in S_n which avoid every pattern in R and we write $I(R)$ to denote the set of all involutions which avoid every pattern in R.

In [1] Egge connected generating functions for various subsets of $I(3412)$ with continued fractions and Chebyshev polynomials of the second kind, and gave a recursive formula for computing them. For instance, he showed [1, Theorem 3.3] that

$$\sum_{\pi \in I(3412)} \prod_{k\geq 1} x_k^{\tau_k(\pi)} = \cfrac{1}{1 - x_1 - \cfrac{x_1^2 x_2}{1 - x_1 x_2^2 x_3 - \cfrac{x_1^2 x_2^5 x_3^4 x_4}{1 - x_1 x_2^4 x_3^6 x_4^4 x_5 - \cdots}}}. \tag{1}$$

Here the nth numerator is $\prod_{i=1}^{2n} x_i^{\binom{2n-2}{i-1}+\binom{2n-1}{i-1}}$, the nth denominator is $1 - \prod_{i=1}^{2n+1} x_i^{\binom{2n}{i-1}}$, and we write $\tau_k(\pi)$ to denote the number of decreasing subsequences of length k in π. Egge also showed [1, Theorem 6.1] that for all $k \geq 1$,

$$\sum_{\sigma \in I(3412,[2k])} x^{|\sigma|} = \frac{U_{k-1}\left(\frac{1-x}{2x}\right)}{xU_k\left(\frac{1-x}{2x}\right)} \tag{2}$$

and

$$\sum_{\sigma \in I(3412,[2k-1])} x^{|\sigma|} = \frac{U_{k-1}\left(\frac{1-x}{2x}\right) + U_{k-2}\left(\frac{1-x}{2x}\right)}{x\left(U_k\left(\frac{1-x}{2x}\right) + U_{k-1}\left(\frac{1-x}{2x}\right)\right)}. \tag{3}$$

Here $U_n(x)$ is the nth Chebyshev polynomial of the second kind, which may be defined by $U_n(\cos t) = \dfrac{\sin((n+1)t)}{\sin t}$. Here we also write $[k]$ to denote the permutation $k(k-1)\ldots 21$; we will use this notation throughout the paper. Egge proved (2) and (3) using the following recursive formula [1, Corollary 5.6] for the generating function $F_\pi(x) = \sum_{\sigma \in I(3412,\pi)} x^{|\sigma|}$, which makes it possible to compute $F_\pi(x)$ for any permutation π:

$$F_\pi(x) = 1 + xF_\beta(x) + x^2 \sum_{i=1}^{k} \left(F_{\overline{\alpha_1 \oplus \cdots \oplus \alpha_i}}(x) - F_{\overline{\alpha_1 \oplus \cdots \oplus \alpha_{i-1}}}(x)\right) F_{\alpha_i \oplus \cdots \oplus \alpha_k}(x). \tag{4}$$

Here the various subscripts of F on the right are the types of certain subsequences of π. For other results concerning one-variable generating functions for pattern-avoiding permutations, continued fractions, and Chebyshev polynomials, see [2, 4, 5] and their references.

We begin this paper by extending Egge's results to certain bivariate generating functions involving the statistic $tc(\pi)$, which is the number of two-cycles in π. For instance, combining (1) with [1, Proposition 3.8], one routinely obtains a continued fraction representation of the generating function $\sum_{\pi \in I(3412)} \prod_{k\geq 1} x_k^{\tau_k(\pi)} y^{tc(\pi)}$. Using Egge's techniques, we expand on this perspective by proving bivariate analogues of (2)–(4). For instance, we show that for all $k \geq 1$,

$$\sum_{\sigma \in I(3412,[2k])} x^{|\sigma|} y^{tc(\sigma)} = \frac{U_{k-1}\left(\frac{1-x}{2x\sqrt{y}}\right)}{x\sqrt{y}\,U_k\left(\frac{1-x}{2x\sqrt{y}}\right)}$$

and

$$\sum_{\sigma\in I(3412,[2k-1])} x^{|\sigma|}y^{tc(\sigma)} = \frac{\sqrt{y}U_{k-1}\left(\frac{1-x}{2x\sqrt{y}}\right)+U_{k-2}\left(\frac{1-x}{2x\sqrt{y}}\right)}{x\sqrt{y}\left(\sqrt{y}U_k\left(\frac{1-x}{2x\sqrt{y}}\right)+U_{k-1}\left(\frac{1-x}{2x\sqrt{y}}\right)\right)}.$$

These results follow from our analogue of (4), which is the following recursive formula for the bivariate generating function $F_\pi(x,y) = \sum_{\sigma\in I(3412,\pi)} x^{|\sigma|}y^{tc(\sigma)}$:

$$F_\pi(x,y) = 1 + xF_\beta(x,y) + x^2y\sum_{i=1}^{k}\left(F_{\overline{\alpha_1\oplus\cdots\oplus\alpha_i}}(x,y) - F_{\overline{\alpha_1\oplus\cdots\oplus\alpha_{i-1}}}(x,y)\right)F_{\alpha_i\oplus\cdots\oplus\alpha_k}(x,y).$$

This result makes it possible to compute $F_\pi(x,y)$ for any permutation π.

In addition to bivariate generating functions for involutions which avoid 3412 and a set of additional patterns, we also study bivariate generating functions for involutions which avoid 3412 and contain a given pattern π exactly once. For any patterns σ_1 and σ_2 and any $n \geq 0$, we write $I_n(\sigma_1;\sigma_2)$ (resp. $I(\sigma_1;\sigma_2)$) to denote the set of involutions of length n (resp. of any length) which avoid σ_1 and contain exactly one subsequence of type σ_2. We first use the recursive structure of $I(3412)$ to obtain recurrence relations for the generating function $G_\pi(x,y) = \sum_{\sigma\in I(3412;\pi)} x^{|\sigma|}y^{tc(\sigma)}$ for certain π. Using these recurrence relations we find $G_\pi(x,y)$ for various π in terms of Chebyshev polynomials of the second kind. For example, we show that for all $k \geq 3$,

$$G_{k\ldots 4213}(x,y) = \frac{x(1-x-x^2y)^2}{(1-x)\left(U_{k+1}\left(\frac{1-x}{2x\sqrt{y}}\right) - x\sqrt{y}U_k\left(\frac{1-x}{2x\sqrt{y}}\right)\right)^2},$$

and we find similar results when 213 is replaced with any permutation of length three.

We then turn our attention to the set $I(;3412)$ of involutions which contain exactly one subsequence of type 3412. Like $I(3412)$, the set $I(;3412)$ has an interesting and useful recursive structure, which we describe. We then use this structure to obtain recurrence relations for the bivariate generating function $P_\pi(x,y) = \sum_{\sigma\in I(\pi;3412)} x^{|\sigma|}y^{tc(\sigma)}$. These recurrence relations enable us to find $P_\pi(x,y)$ for various π in terms of Chebyshev polynomials of the second kind, in spite of the fact that $I(;3412)$ does not appear to be connected with any natural set of lattice paths. For example, we show that for all $k \geq 1$ we have

$$P_{[2k]}(x,y) = \frac{\sum_{j=0}^{k-2} w_{2j+1}(x,y)U_j^2\left(\frac{1-x}{2x\sqrt{y}}\right)}{(1-x^2y)U_k^2\left(\frac{1-x}{2x\sqrt{y}}\right)},$$

where $w_{2j+1}(x,y) = 1 + x - x^{2j+2}y^k - x^{2j+3}y^{j+1}$.

In the final section of the paper we briefly study generating functions for the set of involutions which contain exactly one subsequence of type 3412 and exactly one subsequence of type π.

2 The Recursive Structure of $I(3412)$

In this section we recall the natural recursive structure of $I(3412)$, which was first observed by Guibert [3, Remark 4.28]. To describe this structure, it is convenient to introduce three ways of combining permutations. In particular, suppose $\pi \in S_m$ and $\sigma \in S_n$. We write $\pi \oplus \sigma$ to denote the *direct sum* of π and σ, which is the permutation in S_{m+n} given by

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{if } 1 \le i \le m, \\ \sigma(i-m) + m & \text{if } m+1 \le i \le m+n. \end{cases}$$

We write $\pi \ominus \sigma$ to denote the *skew sum* of π and σ, which is the permutation in S_{m+n} given by

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i) + n & \text{if } 1 \le i \le m, \\ \sigma(i-m) & \text{if } m+1 \le i \le m+n. \end{cases}$$

Finally, we write $\pi * \sigma$ to denote the permutation in S_{m+n+2} given by

$$\pi * \sigma = (1 \ominus \pi \ominus 1) \oplus \sigma.$$

We now describe the recursive structure of $I(3412)$.

Proposition 2.1 *([1, Proposition 2.8])*

(i) *For all $n \ge 1$, the map*

$$\begin{array}{ccc} I_{n-1}(3412) & \longrightarrow & I_n(3412) \\ \pi & \mapsto & 1 \oplus \pi \end{array}$$

is a bijection between $I_{n-1}(3412)$ and the set of involutions in $I_n(3412)$ which begin with 1.

(ii) *For all $n \ge 0$ and all j such that $2 \le j \le n$, the map*

$$\begin{array}{ccc} I_{j-2}(3412) \times I_{n-j}(3412) & \longrightarrow & I_n(3412) \\ (\pi, \sigma) & \mapsto & \pi * \sigma \end{array}$$

is a bijection between $I_{j-2}(3412) \times I_{n-j}(3412)$ and the set of involutions in $I_n(3412)$ which begin with j.

3 Involutions Which Avoid 3412 and Another Pattern

We now turn our attention to certain bivariate generating functions for involutions in $I(3412)$ which avoid a set of additional patterns. We begin by recalling a method of decomposing permutations and a map on permutations.

Definition 3.1 *Fix $n \ge 1$. We call a permutation $\pi \in S_n$* direct sum indecomposable *whenever there do not exist nonempty permutations π_1 and π_2 such that $\pi = \pi_1 \oplus \pi_2$.*

Observe that for every permutation π there exists a unique sequence of direct sum indecomposable permutations $\alpha_1, \ldots, \alpha_k$ such that $\pi = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_k$.

Definition 3.2 *For any permutation π, we define $\overline{\pi}$ as follows.*

1. $\overline{\emptyset} = \emptyset$ *and* $\overline{1} = \emptyset$.

2. *If* $|\pi| \geq 2$ *and there exists a permutation* σ *such that* $\pi = 1 \ominus \sigma \ominus 1$ *then* $\overline{\pi} = \sigma$.

3. *If* $|\pi| \geq 2$, *there exists a permutation* σ *such that* $\pi = 1 \ominus \sigma$, *and* σ *does not end with 1 then* $\overline{\pi} = \sigma$.

4. *If* $|\pi| \geq 2$, *there exists a permutation* σ *such that* $\pi = \sigma \ominus 1$, *and* π *does not begin with* $|\pi|$ *then* $\overline{\pi} = \sigma$.

5. *If* $|\pi| \geq 2$, π *does not begin with* $|\pi|$, *and* π *does not end with 1 then* $\overline{\pi} = \pi$.

Observe that if π and σ are permutations then $1 \ominus \pi \ominus 1$ avoids σ if and only if π avoids $\overline{\sigma}$.

For any involution π, we write $tc(\pi)$ to denote the number of two-cycles in π. For any set of permutations T we write

$$F_T(x, y) = \sum_{\pi \in I(3412, T)} x^{|\pi|} y^{tc(\pi)}.$$

Egge has shown [1, Corollary 5.6] that if $\pi = \alpha_1 \oplus \cdots \oplus \alpha_k$ is a permutation and $\alpha_1, \ldots, \alpha_k$ are direct sum indecomposable then

$$F_\pi(x, 1) = 1 + xF_\beta(x, 1) + x^2 \sum_{i=1}^{k} \left(F_{\overline{\alpha_1 \oplus \cdots \oplus \alpha_i}}(x, 1) - F_{\overline{\alpha_1 \oplus \cdots \oplus \alpha_{i-1}}}(x, 1)\right) F_{\alpha_i \oplus \cdots \oplus \alpha_k}(x, 1), \quad (5)$$

where $\beta = \pi$ if $\alpha_1 \neq 1$ and $\beta = \alpha_2 \oplus \cdots \oplus \alpha_k$ if $\alpha_1 = 1$. Our main result in this section is the following analogue of (5) for $F_\pi(x, y)$.

Theorem 3.3 *Suppose* $\pi = \alpha_1 \oplus \cdots \oplus \alpha_k$ *is a permutation, where* $\alpha_1, \ldots, \alpha_k$ *are direct sum indecomposable. Then*

$$F_\pi(x, y) = 1 + xF_\beta(x, y) + x^2 y \sum_{i=1}^{k} \left(F_{\overline{\alpha_1 \oplus \cdots \oplus \alpha_i}}(x, y) - F_{\overline{\alpha_1 \oplus \cdots \oplus \alpha_{i-1}}}(x, y)\right) F_{\alpha_i \oplus \cdots \oplus \alpha_k}(x, y). \quad (6)$$

Here $\beta = \pi$ *if* $\alpha_1 \neq 1$ *and* $\beta = \alpha_2 \oplus \cdots \oplus \alpha_k$ *if* $\alpha_1 = 1$.

Proof. The set $I(3412, \pi)$ can be partitioned into three sets: the set A_1 containing only the empty permutation, the set A_2 of those involutions which begin with 1, and the set A_3 of those involutions which do not begin with 1.

The set A_1 contributes 1 to the desired generating function.

In view of Proposition 2.1(i), the set A_2 contributes $xF_\beta(x, y)$ to the desired generating function, where $\beta = \pi$ if $\alpha_1 \neq 1$ and $\beta = \alpha_2 \oplus \cdots \oplus \alpha_k$ if $\alpha_1 = 1$.

To obtain the contribution of A_3 to the desired generating function, we first observe that in view of Proposition 2.1(ii), all permutations in A_3 have the form $\sigma_1 * \sigma_2$. Since each α_i is direct sum indecomposable, if $\sigma_1 * \sigma_2$ contains a subsequence of type α_i then that subsequence is entirely contained in either $1 \ominus \sigma_1 \ominus 1$ or σ_2. As a result, the set of involutions which avoid 3412 and π and which do not begin with 1 can be partitioned into sets $B_1, \ldots, B_k$, where B_i is the set of such involutions in which σ_1 contains $\overline{\alpha_1 \oplus \cdots \oplus \alpha_{i-1}}$ but avoids $\overline{\alpha_1 \oplus \cdots \oplus \alpha_i}$.

Since $I_n(3412, \overline{\alpha_1 \oplus \cdots \oplus \alpha_{i-1}}) \subseteq I_n(3412, \overline{\alpha_1 \oplus \cdots \oplus \alpha_i})$, and since the number of two-cycles in $\sigma_1 * \sigma_2$ is one more than the total number of two-cycles in σ_1 and σ_2, the contribution of the set A_3 to the desired generating function is

$$x^2 y \sum_{i=1}^{k} \left(F_{\overline{\alpha_1 \oplus \cdots \oplus \alpha_i}}(x,y) - F_{\overline{\alpha_1 \oplus \cdots \oplus \alpha_{i-1}}}(x,y) \right) F_{\alpha_i \oplus \cdots \oplus \alpha_k}(x,y).$$

Add the contributions of A_1, A_2, and A_3 to obtain (6). $\square$

Theorem 3.3 above is an analogue of [1, Corollary 5.6], which is a special case of [1, Theorem 5.5]. Using Egge's techniques, one can also prove an analogue of [1, Theorem 5.5]. To state this result, we first set some notation.

Definition 3.4 *Let $T = \{\pi_1, \ldots, \pi_m\}$ denote a set of permutations and fix direct sum indecomposable permutations α_j^i, $1 \le i \le m$, $1 \le j \le k_i$, such that $\pi_i = \alpha_1^i \oplus \cdots \oplus \alpha_{k_i}^i$. For all $i_1, \ldots, i_m$ such that $0 \le i_j \le k_j$, let $T_{i_1,\ldots,i_m}^{right} = \{\alpha_{i_1}^1 \oplus \cdots \oplus \alpha_{k_1}^1, \ldots, \alpha_{i_m}^m \oplus \cdots \oplus \alpha_{k_m}^m\}$. For any subset $Y \subseteq \{1, \ldots, m\}$, set*

$$T_Y = \bigcup_{j \in Y} \{\overline{\alpha_1^j \oplus \cdots \oplus \alpha_{i_j - 1}^j}\} \bigcup_{j \notin Y, 1 \le j \le m} \{\overline{\alpha_1^j \oplus \cdots \oplus \alpha_{i_j}^j}\}.$$

Theorem 3.5 *With reference to Definition 3.4,*

$$F_T(x,y) = 1 + xF_{\beta(T)}(x,y) + x^2 y \sum_{i_1,\ldots,i_m=1}^{k_1,\ldots,k_m} \left(\sum_{Y \subseteq \{1,2,\ldots,m\}} (-1)^{|Y|} F_{T_Y}(x,y) \right) F_{T_{i_1,\ldots,i_m}^{right}}(x,y).$$

Here $\beta(\pi_i) = \pi_i$ if $\alpha_1^i \neq 1$, $\beta(\pi_i) = \alpha_2^i \oplus \cdots \oplus \alpha_{k_i}^i$ if $\alpha_1^i = 1$, and $\beta(T)$ is the set of permutations obtained by applying β to every element of T.

We omit the proof of Theorem 3.5 for the sake of brevity.

Using (6), it is possible to find $F_\pi(x,y)$ for any permutation π. For the remainder of this section we concentrate on a family of permutations connected with a certain symmetry conjecture. We express all of our generating functions in terms of Chebyshev polynomials of the second kind, so we begin by recalling these polynomials.

Definition 3.6 *For all n we write $U_n(x)$ to denote the n*th Chebyshev polynomial of the second kind, *which is defined by $U_n(x) = 0$ for $n < 0$ and $U_n(\cos t) = \dfrac{\sin((n+1)t)}{\sin t}$ for $n \ge 0$. These polynomials satisfy*

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \qquad (n \neq 0). \tag{7}$$

We will often use a certain specialization of $U_n(x)$, which is defined for all n by

$$V_n = V_n(x,y) = U_n\left(\frac{1-x}{2x\sqrt{y}}\right).$$

Observe that by (7) we have

$$x\sqrt{y}V_n(x,y) = (1-x)V_{n-1}(x,y) - x\sqrt{y}V_{n-2}(x,y). \tag{8}$$

Turning our attention to $F_\pi(x,y)$, we now consider permutations of the forms $[j] \ominus \pi \ominus [j]$, $[j] \ominus \pi$, and $\pi \ominus [j]$.

Proposition 3.7 *For any permutation π we have*

$$F_{1\ominus\pi\ominus 1}(x,y)=\frac{1}{1-x-x^2yF_\pi(x,y)}. \tag{9}$$

Moreover, if $F_\pi(x,y)=\frac{f_0(x,y)}{f_1(x,y)}$ for polynomials f_0 and f_1 then for all $j\ge 1$,

$$F_{[j]\ominus\pi\ominus[j]}(x,y)=\frac{f_1(x,y)V_{j-1}(x,y)-x\sqrt{y}f_0(x,y)V_{j-2}(x,y)}{x\sqrt{y}f_1(x,y)V_j(x,y)-x^2yf_0(x,y)V_{j-1}(x,y)}. \tag{10}$$

Proof. To prove (9), replace π with $1\ominus\pi\ominus 1$ in (6), use the fact that $1\ominus\pi\ominus 1$ is direct sum indecomposable and $\beta=1\ominus\pi\ominus 1$ to simplify the result, and solve for $F_{1\ominus\pi\ominus 1}(x,y)$.

To prove (10) we argue by induction on j. When $j=1$ the result is immediate from (9), so we assume $j\ge 2$ and the result holds for $j-1$. Replace π with $[j-1]\ominus\pi\ominus[j-1]$ in (9), use induction to eliminate $F_{[j-1]\ominus\pi\ominus[j-1]}(x,y)$ on the right, and use (8) to simplify the result and obtain (10). $\Box$

Arguing as in the proof of Proposition 3.7, one can show that if π does not end with 1 then (9) and (10) hold when $1\ominus\pi\ominus 1$ (resp. $[j]\ominus\pi\ominus[j]$) is replaced with $1\ominus\pi$ (resp. $[j]\ominus\pi$). Similarly, one can show that if π does not begin with $|\pi|$ then (9) and (10) hold when $1\ominus\pi\ominus 1$ (resp. $[j]\ominus\pi\ominus[j]$) is replaced with $\pi\ominus 1$ (resp. $\pi\ominus[j]$).

With Proposition 3.7 and its analogues in hand, we are ready to compute $F_\pi(x,y)$ for various π. We begin with $\pi=[k]$, which is the decreasing permutation of length k.

Theorem 3.8 *For all $k\ge 1$ we have*

$$F_{[2k]}(x,y)=\frac{V_{k-1}(x,y)}{x\sqrt{y}V_k(x,y)} \tag{11}$$

and

$$F_{[2k-1]}(x,y)=\frac{\sqrt{y}V_{k-1}(x,y)+V_{k-2}(x,y)}{x\sqrt{y}\left(\sqrt{y}V_k(x,y)+V_{k-1}(x,y)\right)}. \tag{12}$$

Proof. To prove (11), set $\pi=\emptyset$ and $j=k$ in (10) and use the fact that $F_\pi(x,y)=\frac{0}{1}$.

The proof of (12) is similar to the proof of (11). $\Box$

We now focus our attention on the case in which π is a layered permutation.

Definition 3.9 *Fix $n\ge 1$ and let $l_1,l_2,\ldots,l_m$ denote a sequence such that $l_i\ge 1$ for $1\le i\le m$ and $\sum_{i=1}^m l_i=n$. We write $[l_1,l_2,\ldots,l_m]$ to denote the permutation in S_n given by*

$$[l_1,l_2,\ldots,l_m]=[l_1]\oplus\cdots\oplus[l_m].$$

We call a permutation layered *whenever it has the form $[l_1,\ldots,l_m]$ for some sequence $l_1,\ldots,l_m$.*

Observe that if $m\ge 2$ then $\overline{[l_1,\ldots,l_m]}=[l_1,\ldots,l_m]$. In view of (6), (11), and (12), the generating function $F_{[l_1,\ldots,l_m]}(x,y)$ can be expressed in terms of Chebyshev polynomials of the second kind for any layered permutation $[l_1,\ldots,l_m]$. To do this for $m=2$ we will use the following well-known identity for Chebyshev polynomials.

Lemma 3.10 *For all $k, l \geq -1$ and all $w \geq 0$ we have*

$$U_{k+w}U_{l+w} - U_kU_l = U_{w-1}U_{k+l+w+1}. \tag{13}$$

It will also be useful to record the recurrence relations for $F_{[k,l]}(x, y)$.

Lemma 3.11 *For all $k \geq 2$ and all $l \geq 1$ we have*

$$F_{[1,l]}(x, y) = \frac{1 + xF_{[l]}(x, y)}{1 - x^2yF_{[l]}(x, y)} \tag{14}$$

and

$$F_{[k,l]}(x, y) = \frac{1 - x^2yF_{[k-2]}(x, y)F_{[l]}(x, y)}{1 - x - x^2yF_{[k-2]}(x, y) - x^2yF_{[l]}(x, y)}. \tag{15}$$

Proof. To prove (14), set $\pi = [1, l]$ in (6) and solve the resulting equation for $F_{[1,l]}(x, y)$.

The proof of (15) is similar to the proof of (14). $\Box$

We now compute $F_{[k,l]}(x, y)$ when k and l are not both odd.

Theorem 3.12 *For all $k, l \geq 1$ such that k and l are not both odd we have*

$$F_{[k,l]}(x, y) = F_{[k+l]}(x, y). \tag{16}$$

Proof. We consider four cases: $k = 1$ and l is even, k and l are both even, k is even and l is odd, and $k > 1$ is odd and l is even. All four cases are similar, so we only give the details for the case in which k is even and l is odd.

In (15) replace k with $2k$ and l with $2l - 1$, use (11) and (12) to write the result in terms of Chebyshev polynomials, clear denominators, and use (8) in the resulting denominator to obtain

$$F_{[2k,2l-1]}(x, y) = \frac{V_{k-1}\left(\sqrt{y}V_l + V_{l-1}\right) - V_{k-2}\left(\sqrt{y}V_{l-1} + V_{l-2}\right)}{x\sqrt{y}\left(V_k\left(\sqrt{y}V_l + V_{l-1}\right) - V_{k-1}\left(\sqrt{y}V_{l-1} + V_{l-2}\right)\right)}.$$

Now apply (13) to the numerator and denominator to obtain the right side of (12) with k replaced by $k + l$, as desired. $\Box$

Next we compute $F_{[k,l]}(x, y)$ when k and l are both odd.

Theorem 3.13 *For all $k, l \geq 1$ we have*

$$F_{[2k-1,2l-1]}(x, y) = \frac{yV_{k+l-1} + 2\sqrt{y}V_{k+l-2} + V_{k+l-3}}{x\sqrt{y}(yV_{k+l} + 2\sqrt{y}V_{k+l-1} + V_{k+l-2})}. \tag{17}$$

Proof. We consider two cases: $k = 1$ and $k > 1$. These are similar, so we only give details for the case in which $k > 1$.

In (15) replace k with $2k - 1$ and l with $2l - 1$, use (12) and (8) to write the result in terms of Chebyshev polynomials, clear denominators, and use (8) in the denominator to obtain

$$F_{[2k-1,2l-1]}(x, y) = \frac{(\sqrt{y}V_{k-1} + V_{k-2})(\sqrt{y}V_l + V_{l-1}) - (\sqrt{y}V_{k-2} + V_{k-3})(\sqrt{y}V_{l-1} + V_{l-2})}{x\sqrt{y}\left((\sqrt{y}V_k + V_{k-1})(\sqrt{y}V_l + V_{l-1}) - (\sqrt{y}V_{k-1} + V_{k-2})(\sqrt{y}V_{l-1} + V_{l-2})\right)}.$$

Group terms in the numerator and denominator by powers of y and apply (13) repeatedly to obtain (17), as desired. □

When $m \geq 3$ the generating function $F_{[l_1,\ldots,l_m]}(x, y)$ does not reduce quite as nicely as it does when $m = 2$. For example, using the same techniques as in the proof of Theorem 3.12 one can prove that for all $k_1, k_2, k_3 \geq 1$,

$$F_{[2k_1,2k_2,2k_3]}(x, y) = \frac{V_{k_1+k_2+k_3}V_{k_1+k_2+k_3-1} + V_{k_1+k_2-1}V_{k_1+k_3-1}V_{k_2+k_3-1}}{x\sqrt{y}V_{k_1+k_2}V_{k_1+k_3}V_{k_2+k_3}}. \tag{18}$$

Nevertheless, (16), (17), and (18) suggest the following conjecture.

Conjecture 3.14 *For all $m \geq 1$ and all $l_1, \ldots, l_m \geq 1$, the generating function $F_{[l_1,\ldots,l_m]}(x, y)$ is symmetric in $l_1, \ldots, l_m$.*

We have verified this conjecture in the case $m = 3$ for $l_i \leq 24$ and in the case $m = 4$ for $l_i \leq 20$ using a Maple program.

Egge has previously conjectured [1, Conjecture 6.9] that $F_{[l_1,\ldots,l_m]}(x, 1)$ is symmetric in $l_1, \ldots, l_m$; clearly [1, Conjecture 6.9] follows from Conjecture 3.14.

4 Involutions Which Avoid 3412 and Contain Another Pattern Exactly Once

In [1] Egge finds the generating function for the involutions in $I(3412)$ which contain exactly r subsequences of type $[k]$. Using similar techniques, it is routine to find the corresponding bivariate generating function in which the second variable records the number of two-cycles. In this section we study bivariate generating functions for involutions which avoid 3412 and contain another pattern exactly once. In particular, for any permutation π we write

$$G_\pi(x, y) = \sum_{\sigma \in I(3412;\pi)} x^{|\sigma|}y^{tc(\sigma)}.$$

In this section we give recurrence relations for $G_\pi(x, y)$ for certain π and we use these recurrence relations to give $G_\pi(x, y)$ in terms of Chebyshev polynomials for some of these π. We begin with the case in which π begins with $|\pi|$ and ends with 1.

Proposition 4.1 *For any permutation π which begins with $|\pi|$ and ends with 1 we have*

$$G_{1\ominus\pi\ominus1}(x, y) = x^2yG_\pi(x, y) \left(F_{1\ominus\pi\ominus1}(x, y)\right)^2. \tag{19}$$

Proof. To prove (19), first observe that the set $I(3412; 1\ominus\pi\ominus1)$ can be partitioned into two sets: the set A_1 of those involutions which begin with 1 and the set A_2 of those involutions which do not begin with 1.

In view of Proposition 2.1(i), the set A_1 contributes $xG_{1\ominus\pi\ominus1}(x, y)$ to the desired generating function.

In view of Proposition 2.1(ii), every permutation in A_2 has the form $\sigma_1*\sigma_2$. Since $1\ominus\pi\ominus1$ is direct sum indecomposable, A_2 can be partitioned into two sets: the set B_1 of involutions in which $1\ominus\pi\ominus1$ occurs in $1\ominus\sigma_1\ominus1$ and the set B_2 of involutions in which $1\ominus\pi\ominus1$ occurs in σ_2. The set B_1 contributes $x^2yG_\pi(x,y)F_{1\ominus\pi\ominus 1}(x,y)$ to the desired generating function and the set B_2 contributes $x^2yG_{1\ominus\pi\ominus 1}(x,y)F_\pi(x,y)$.

Add the contributions of A_1, B_1, and B_2 and solve the resulting equation for $G_{1\ominus\pi\ominus 1}(x,y)$ to obtain

$$G_{1\ominus\pi\ominus 1}(x,y) = \frac{x^2yG_\pi(x,y)F_{1\ominus\pi\ominus 1}(x,y)}{1-x-x^2yF_\pi(x,y)}.$$

Now (19) follows from (9). $\square$

Arguing as in the proof of Proposition 4.1, one can show that if π begins with $|\pi|$ and does not end with 1 then (19) holds when $1\ominus\pi\ominus 1$ is replaced with $1\ominus\pi$. Similarly, one can show that if π ends with 1 and does not begin with $|\pi|$ then (19) holds when $1\ominus\pi\ominus 1$ is replaced with $\pi\ominus 1$.

For the remainder of this section we use Proposition 4.1 and its analogues to compute $G_\pi(x,y)$ for various π. In each case we express $G_\pi(x,y)$ in terms of Chebyshev polynomials of the second kind. We begin with $G_\pi(x,y)$ for various π of length 2 or 3.

Proposition 4.2 *We have*

$$G_{12}(x,y) = \frac{x^2}{1-x^2y}, \tag{20}$$

$$G_{312}(x,y) = G_{231}(x,y) = \frac{x^4y(1-x^2y)^2}{((1-x)(1-x^2y)-x^2y(1+x))^2}, \tag{21}$$

and

$$G_{213}(x,y) = G_{132}(x,y) = \frac{x^3y}{1-x-x^2y}. \tag{22}$$

Proof. To prove (20), first observe that if a permutation contains exactly one subsequence of type 12 then it has the form $n\ (n-1)\dots(i+2)\ i\ (i+1)\ (i-1)\ \dots 21$ for some i, $1\le i\le n-1$. Now observe that permutations of this form are involutions if and only if n is even and $i=\frac{n}{2}$. Now (20) follows.

To prove (21), first observe that 312 is the reverse complement of 231, and the reverse complement operation preserves the cycle structure, so $G_{312}(x,y) = G_{231}(x,y)$. To find $G_{312}(x,y)$, first argue as in the proof of Proposition 4.1 to find

$$G_{312}(x,y) = xG_{312}(x,y) + x^2yH_{312,12}(x,y)F_{312}(x,y) + x^2y\frac{1+x}{1-x^2y}G_{312}(x,y), \tag{23}$$

where $H_{312,12}(x,y)$ is the bivariate generating function for the involutions which avoid 312 and contain exactly one subsequence of type 12. To obtain $H_{312,12}(x,y)$, observe that if a permutation avoids 312 and contains exactly one subsequence of type 12 then it has the form $(n-1)n(n-2)(n-3)\cdots 21$. Among these, only 12 is an involution, so $H_{312,12}(x,y)=x^2$. Use this to eliminate $H_{312,12}(x,y)$ in (23), use the fact that

$$F_{312}(x,y) = \frac{1-x^2y}{(1-x)(1-x^2y)-x^2y(1+x)}$$

to eliminate $F_{312}(x,y)$, and solve the resulting equation for $G_{312}(x,y)$ to complete the proof of (21).

The proof of (22) is similar to the proof of (21), using

$$G_{213}(x,y) = xG_{213}(x,y) + x^3y + x^2yG_{213}(x,y).$$

□

Using our expressions for $G_{312}(x,y)$ and $G_{231}(x,y)$, we obtain expressions for $G_{[k]\ominus 231}(x,y)$ and $G_{[k]\ominus 12}(x,y)$.

Proposition 4.3 *For all $k \geq 1$ we have*

$$G_{[k]\ominus 12}(x,y) = G_{[k-1]\ominus 231}(x,y) = \frac{x^2(1-x^2y)^2}{\left((1-x^2y)V_k - x\sqrt{y}(1+x)V_{k-1}\right)^2}. \tag{24}$$

Proof. The fact that $G_{[k]\ominus 12}(x,y) = G_{[k-1]\ominus 231}(x,y)$ follows by induction on k from (21) and the analogue of (19) in which $1 \ominus \pi \ominus 1$ is replaced with $1 \ominus \pi$.

To complete the proof of (24) we show that $G_{[k]\ominus 12}(x,y)$ is equal to the quantity on the right for $k \geq 0$. Arguing by induction on k, first observe that when $k = 1$ the result is immediate from (21). Now suppose $k \geq 2$ and the result holds for $k-1$. Set $\pi = [k-1] \ominus 12$ in the analogue of (19) in which $1 \ominus \pi \ominus 1$ is replaced with $1 \ominus \pi$. Using the analogue of (10) in which $[j] \ominus \pi \ominus [j]$ is replaced with $[j] \ominus \pi$ and the fact that $F_{12}(x,y) = \frac{1+x}{1-x^2y}$ we have

$$F_{[k]\ominus 12}(x,y) = \frac{(1-x^2y)V_{k-1} - x\sqrt{y}(1+x)V_{k-2}}{x\sqrt{y}(1-x^2y)V_k - x^2y(1+x)V_{k-1}}. \tag{25}$$

Use this to eliminate $F_{[k]\ominus 12}(x,y)$ and use induction to eliminate $G_{[k-1]\ominus 12}(x,y)$ and obtain the quantity on the right side of (24), as desired. □

We conclude this section by obtaining expressions for $G_{[k]\ominus 213}(x,y)$ and $G_{[k]\ominus 132}(x,y)$.

Proposition 4.4 *For all $k \geq 1$ we have*

$$G_{[k]\ominus 213}(x,y) = G_{[k]\ominus 132}(x,y) = \frac{x(1-x-x^2y)^2}{(1-x)\left(V_{k+1} - x\sqrt{y}V_k\right)^2}.$$

Proof. To derive $G_{[k]\ominus 213}(x,y)$, first argue as in the proof of (21) to obtain

$$G_{4213}(x,y) = xG_{4213}(x,y) + x^2y \cdot \frac{x^3y}{1-x} \cdot F_{4213}(x,y) + x^2yF_{213}(x,y)G_{4213}(x,y).$$

Solve this equation for $G_{4213}(x,y)$ and use the fact that

$$F_{[k]\ominus 132}(x,y) = F_{[k]\ominus 213}(x,y) = \frac{V_k - x\sqrt{y}V_{k-1}}{x\sqrt{y}(V_{k+1} - x\sqrt{y}V_k)} \qquad (k \geq 0)$$

to simplify the result. To find $G_{[k]\ominus 213}(x,y)$, use the analogue of (19) in which $1 \ominus \pi \ominus 1$ is replaced with $1 \ominus \pi$ and argue by induction on k.

The derivation of $G_{[k]\ominus 132}(x,y)$ is similar to the derivation of $G_{[k]\ominus 213}(x,y)$. □

Using the same techniques one can also compute $G_{123}(x,y)$; we leave this derivation as an exercise for the diligent reader.

In view of Conjecture 3.14 one might conjecture that $G_{[l_1,\ldots,l_m]}(x,y)$ is symmetric in $l_1,\ldots,l_m$. However, it is not difficult to verify that the coefficient of x^5 in $G_{[2,1,1]}(x,y)$ is y and the coefficient of x^5 in $G_{[1,2,1]}(x,y)$ is 0, from which it follows that $G_{[2,1,1]}(x,y) \neq G_{[1,2,1]}(x,y)$, and even that $G_{[2,1,1]}(x,1) \neq G_{[1,2,1]}(x,1)$.

5 Involutions Which Contain 3412 Exactly Once and Avoid Another Pattern

In this section we study involutions which contain exactly one subsequence of type 3412. We begin by describing the recursive structure of the set of such involutions. We then use this recursive structure to find the bivariate generating function

$$P_\pi(x,y) = \sum_{\sigma \in I(\pi;3412)} x^{|\sigma|} y^{tc(\sigma)}$$

for various permutations π. Often we obtain relatively simple expressions for $P_\pi(x,y)$ in terms of Chebyshev polynomials of the second kind. We start with the notion of a crossing in an involution.

Definition 5.1 *Let π denote an involution. A* crossing *in π is a sequence $i < j < k < l$ such that $\pi(i) = k$ and $\pi(j) = l$. Each crossing has a corresponding subsequence in π, given by $\pi(i)\pi(j)\pi(k)\pi(l)$.*

Observe that in an involution every crossing produces a subsequence of type 3412, but some subsequences of type 3412 do not have corresponding crossings. For example, 563412 has one crossing and three subsequences of type 3412. However, as we show next, involutions which avoid 3412 are exactly those involutions which have no crossings.

Proposition 5.2 *Suppose π is an involution. Then π avoids 3412 if and only if π has no crossings.*

Proof. ($\Longrightarrow$) Suppose $\pi \in I(3412)$. Observe that if $abcd$ is a crossing then its corresponding subsequence has type 3412. But π avoids 3412, so π has no crossings.

($\Longleftarrow$) Suppose π is an involution and $\pi(a)\ \pi(b)\ \pi(c)\ \pi(d)$ is a subsequence of type 3412; we show π contains a crossing. We first consider the case in which $\pi(a) \le a$. Since $\pi(c) < \pi(a) \le a$ and $\pi(d) < \pi(a) \le a$, we find that $\pi(c) < c$ and $\pi(d) < d$. It follows that $\pi(c)\pi(d)cd$ is a crossing, since $\pi(c) < \pi(d)$. Now suppose $\pi(a) > a$, and by a similar argument we find $\pi(b) > b$. It follows that $ab\pi(a)\pi(b)$ is a crossing, since $\pi(a) < \pi(b)$. □

We now consider crossings in involutions which contain exactly one subsequence of type 3412.

Corollary 5.3 *Suppose π is an involution with exactly one subsequence of type 3412, given by $\pi(a)\pi(b)\pi(c)\pi(d)$. Then π has exactly one crossing, which is $abcd$.*

Proof. By Proposition 5.2, the involution π has at least one crossing. Moreover, each crossing corresponds to a subsequence of type 3412, so π has at most one crossing. It follows that π has exactly one crossing. In addition, the subsequence corresponding to this crossing has type 3412, so the subsequence must be $\pi(a)\pi(b)\pi(c)\pi(d)$. It follows that the crossing is $abcd$. □

We now describe the recursive structure of $I(;3412)$.

Theorem 5.4 *Fix $\pi \in I_n(;3412)$. Then exactly one of the following holds.*

(i) *π begins with 1.*

(ii) *π does not begin with 1, 1 is not an element of the subsequence of π of type 3412, and this subsequence lies between $\pi(1)$ and 1.*

(iii) *π does not begin with 1, 1 is not an element of the subsequence of π of type 3412, and this subsequence lies entirely to the right of 1.*

(iv) *1 is an element of the subsequence of π of type 3412.*

Proof. Observe it is sufficient to prove both of the following.

(A) If π does not begin with 1 and 1 is not an element of the subsequence of type 3412 then this subsequence lies between $\pi(1)$ and 1 or it lies entirely to the right of 1.

(B) If 1 is an element of the subsequence of π of type 3412 then π does not begin with 1.

To prove (A), suppose by way of contradiction that $\pi(a)\pi(b)\pi(c)\pi(d)$ is the subsequence of type 3412 and 1 lies between $\pi(a)$ and $\pi(d)$. By Corollary 5.3 we have $\pi(a) = c$ and $\pi(b) = d$. It follows that at least one of $\pi(1)a1c$ and $\pi(1)b1d$ is a crossing in π. But $abcd$ is the unique crossing in π, so $c = \pi(a) = 1$, which contradicts the fact that 1 is not an element of the subsequence of π of type 3412.

To prove (B), observe that if 1 is an element of the subsequence of type 3412 in π then it is the third element of this subsequence, so it cannot be the first element of π. □

Definition 5.5 *For any permutations π_1, π_2, and π_3 we write $\pi_1 \otimes \pi_2 \otimes \pi_3$ to denote the permutation in $S_{|\pi_1|+|\pi_2|+|\pi_3|+4}$ which is given by*

$$(\pi_1 \otimes \pi_2 \otimes \pi_3)(i) = \begin{cases} |\pi_1| + |\pi_2| + 3 & \text{if } i = 1 \\ \pi_1(i-1) + 1 & \text{if } 2 \le i \le |\pi_1| + 1 \\ |\pi_1| + |\pi_2| + |\pi_3| + 4 & \text{if } i = |\pi_1| + 2 \\ \pi_2(i - |\pi_1| - 2) + |\pi_1| + 2 & \text{if } 3 \le i - |\pi_1| \le |\pi_2| + 2 \\ 1 & \text{if } i = |\pi_1| + |\pi_2| + 3 \\ \pi_3(i - |\pi_1| - |\pi_2| - 3) + |\pi_1| + |\pi_2| + 3 & \text{if } 4 \le i - |\pi_1| - |\pi_2| \le +|\pi_3| + 3 \\ 1 + |\pi_1| & \text{if } i = |\pi_1| + |\pi_2| + |\pi_3| + 4 \end{cases}.$$

Theorem 5.6 (i) *For all $n \ge 1$, the map*

$$\begin{array}{rcl} I_{n-1}(;3412) & \longrightarrow & I_n(;3412) \\ \pi & \mapsto & 1 \oplus \pi \end{array}$$

is a bijection between $I_{n-1}(;3412)$ and the set of involutions in $I_n(;3412)$ which begin with 1.

(ii) *For all $n \ge 1$ and all j such that $2 \le j \le n$, the map*

$$\begin{array}{rcl} I_{j-2}(;3412) \times I_{n-j}(3412) & \longrightarrow & I_n(;3412) \\ (\pi_1, \pi_2) & \mapsto & \pi_1 * \pi_2 \end{array}$$

is a bijection between $I_{j-2}(;3412) \times I_{n-j}(3412)$ and the set of involutions in $I_n(;3412)$ which do not begin with 1, in which 1 is not an element of the subsequence of type 3412, and in which this subsequence lies between $\pi(1)$ and 1.

(iii) *For all $n \geq 1$ and all j such that $2 \leq j \leq n$, the map*

$$\begin{array}{rcl} I_{j-2}(3412) \times I_{n-j}(;3412) & \longrightarrow & I_n(;3412) \\ (\pi_1, \pi_2) & \mapsto & \pi_1 * \pi_2 \end{array}$$

is a bijection between $I_{j-2}(3412) \times I_{n-j}(3412)$ and the set of involutions in $I_n(;3412)$ which do not begin with 1, in which 1 is not an element of the subsequence of type 3412, and in which this subsequence lies entirely to the right of 1.

(iv) *For each $k \geq 0$, set $[k] = k\ k-1\ \ldots 21$. For all $n \geq 1$ and all $j, k, m \geq 0$ such that $0 \leq j + k + m \leq n - 4$, the map*

$$I_j(3412) \times I_k(3412) \times I_m(3412) \longrightarrow I_n(;3412)$$

$$(\pi_1, \pi_2, \pi_3) \mapsto (\pi_1 \otimes [n - j - k - m - 4] \otimes \pi_2) \oplus \pi_3$$

is a bijection between $I_j(3412) \times I_k(3412) \times I_m(3412)$ and the set of involutions in $I_n(;3412)$ in which 1 is an element of the subsequence of type 3412.

Proof. (i) This is immediate from the fact that if π begins with 1 then 1 cannot be an element of a subsequence of type 3412 in π.

(ii) Observe that if the subsequence of type 3412 lies between $\pi(1)$ and 1 then the elements of π to the left of 1 are $2, 3, \ldots, \pi(1)$. Now (ii) follows.

(iii) This is similar to the proof of (ii).

(iv) It is routine to verify that every permutation of the given form contains exactly one subsequence of type 3412, and that 1 is an element of this subsequence.

To show that the given map is one-to-one and onto, suppose $\pi \in I_n(;3412)$ and that 1 is an element of the subsequence of type 3412 in π. Let $\pi(a)\pi(b)\pi(c)\pi(d)$ denote this subsequence. Then there exist unique sequences $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4$ which avoid 3412 such that $\pi = \sigma_0\pi(a)\sigma_1\pi(b)\sigma_2\pi(c)\sigma_3\pi(d)\sigma_4$. Since 1 is an element of $\pi(a)\pi(b)\pi(c)\pi(d)$ and this sequence has type 3412, we have $\pi(c) = 1$. By Corollary 5.3 we have $\pi(1) = a$, so σ_0 is empty.

Now set $j = |\sigma_1|$, $k = |\sigma_3|$, and $m = |\sigma_4|$, so that $|\sigma_2| = n-j-k-m-4$. Since $abcd$ is the only crossing in π, the entries of σ_1 are $2, \ldots, j+1$, the entries of σ_2 are $j+3, \ldots, n-k-m-2$, the entries of σ_3 are $n - k - m, \ldots, n - m - 1$, and the entries of σ_4 are $n - m + 1, \ldots, n$. Moreover, if σ_2 has a subsequence $\pi(e)\pi(f)$ of type 12 then $\pi(a)\pi(b)\pi(e)\pi(f)$ will have type 3412. It follows that σ_2 has type $[n - j - k - m - 4]$. Combining these observations, we find there exist unique permutations π_1, π_2, π_3 such that $|\pi_1| = j$, $|\pi_2| = k$, $|\pi_3| = m$, and $\pi = (\pi_1 \otimes [n - j - k - m - 4] \otimes \pi_2) \oplus \pi_3$. It follows that the given map is a bijection, as desired. $\square$

Using Theorem 5.6, we find the generating function for those involutions which contain exactly one subsequence of type 3412.

Proposition 5.7 *We have*

$$P_\emptyset(x, y) = \frac{(1+x)(4 - 12x + 12x^2 - 12x^2y - 4x^3 + 12x^3y)}{8x^2y(1 - x^2y)\sqrt{(x-1)^2 - 4x^2y}} - \frac{(1+x)(4 - 8x + 4x^2 - 4x^2y)}{8x^2y(1 - x^2y)}. \tag{26}$$

Proof. To prove (26), first use Theorems 5.4 and 5.6 to find that

$$P_\emptyset(x,y) = xP_\emptyset(x,y) + 2x^2yP_\emptyset(x,y)F_\emptyset(x,y) + \frac{x^4y^2(1+x)(F_\emptyset(x,y))^3}{1-x^2y}.$$

Now use the fact that

$$F_\emptyset(x,y) = \frac{1-x-\sqrt{(x-1)^2-4x^2y}}{2x^2y}$$

to eliminate $F_\emptyset(x,y)$ and solve the resulting equation for $P_\emptyset(x,y)$ to obtain (26). □

Theorem 5.6 enables us to obtain $P_{1\ominus\pi\ominus 1}(x,y)$ for many π. We begin with the case in which π is decreasing.

Proposition 5.8 *For all $k \geq 1$ we have*

$$P_{[2k]}(x,y) = \frac{\sum_{j=0}^{k-2} w_{2j+1}(x,y)V_j^2(x,y)}{(1-x^2y)V_k^2(x,y)} \tag{27}$$

and

$$P_{[2k-1]}(x,y) = \frac{\sum_{j=0}^{k-2} w_{2j}(x,y)\left(\sqrt{y}V_j(x,y)+V_{j-1}(x,y)\right)^2}{(1-x^2y)\left(\sqrt{y}V_k(x,y)+V_{k-1}(x,y)\right)^2}. \tag{28}$$

Here $w_{2k}(x,y) = 1+x-x^{2k+1}y^k - x^{2k+2}y^{k+1}$ and $w_{2k-1}(x,y) = 1+x-x^{2k}y^k-x^{2k+1}y^k$.

Proof. To begin, use Theorems 5.4 and 5.6 to find that

$$\begin{aligned} &P_{[k]}(x,y) = \\ &\quad xP_{[k]}(x,y) + x^2yP_{[k-2]}(x,y)F_{[k]}(x,y) + x^2yF_{[k-2]}(x,y)P_{[k]}(x,y) \\ &\quad + x^4y^2\frac{w_{k-3}(x,y)}{1-x^2y}\left(F_{[k-2]}(x,y)\right)^2 F_{[k]}(x,y). \end{aligned}$$

Now solve this equation for $P_{[k]}(x,y)$, use Theorem 3.8 to eliminate $F_{[k]}(x,y)$ and $F_{[k-2]}(x,y)$ and argue by induction on k to obtain (27) and (28). □

Proposition 5.8 enables us to find $|I_n([k];3412)|$ for various k. For instance, for all $n \geq 4$ we find

$$|I_n(321;3412)| = \frac{n-3}{5}F_{n-1} + \frac{n-1}{5}F_{n-3}$$

and

$$|I_n(4321;3412)| = 2^{n-5}(3n-10).$$

Here F_n is the nth Fibonacci number, which is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Next we give a recursive procedure for computing $P_{1\ominus\pi\ominus 1}(x,y)$ when π is direct sum indecomposable and is not of the form $[k]$. This result is the analogue of (9) for $P_{1\ominus\pi\ominus 1}(x,y)$.

Proposition 5.9 *For any direct sum indecomposable permutation π which is not of the form $[k]$ we have*

$$P_{1\ominus\pi\ominus 1}(x,y) = x^2y\left(F_{1\ominus\pi\ominus 1}(x,y)\right)^2\left(P_\pi(x,y) + \frac{x^2y(1+x)}{1-x^2y}\left(F_\pi(x,y)\right)^2\right). \tag{29}$$

Proof. To prove (29), use Theorems 5.4 and 5.6, the fact that π is direct sum indecomposable, and the fact that π is not of the form $[k]$ to find that

$$\left(1 - x - x^2 y F_\pi(x,y)\right) P_{1\ominus\pi\ominus 1}(x,y) = \\ x^2 y F_{1\ominus\pi\ominus 1}(x,y) P_\pi(x,y) + \frac{x^4 y^2 (1+x)}{1 - x^2 y} \left(F_\pi(x,y)\right)^2 F_{1\ominus\pi\ominus 1}(x,y).$$

Now (29) follows from (9). □

As some of our later results will indicate, the analogue of Proposition 5.9 in which $1\ominus\pi\ominus 1$ is replaced with $1 \ominus \pi$ does not hold in general, even if π does not end with 1. Similarly, the analogue of Proposition 5.9 in which $1 \ominus \pi \ominus 1$ is replaced with $\pi \ominus 1$ does not hold in general, even if π does not begin with $|\pi|$.

In addition to allowing one to find $P_{1\ominus\pi\ominus 1}(x,y)$, Theorem 5.6 also enables one to find $P_\sigma(x,y)$ when σ is not of the form $1 \ominus \pi \ominus 1$. For instance, next we obtain $P_\pi(x,y)$ for any π of length 3.

Proposition 5.10 *We have*

$$P_{123}(x,y) = \frac{x^4 y^2 (1 + 3x + 3x^2 + x^2 y + 3x^3 y + x^4 y^2)}{(1 - x^2 y)^3},$$

$$P_{132}(x,y) = P_{213}(x,y) = \frac{x^4 y^2 (1+x)}{(1 - x - x^2 y)(1 - x^2 y)},$$

and

$$P_{231}(x,y) = P_{312}(x,y) = 0.$$

Proof. The last line is immediate, since a permutation which contains a subsequence of type 3412 must also contain subsequences of type 231 and 312.

The fact that $P_{132}(x,y) = P_{213}(x,y)$ is immediate, since 213 is the reverse complement of 132 and this operation preserves the cycle structure.

To obtain $P_{132}(x,y)$, use Theorem 5.6 to find that

$$P_{132}(x,y) = x^2 y \left(\frac{1}{1-x}\right) P_{132}(x,y) + x^4 y^2 \cdot \frac{1+x}{1 - x^2 y} \cdot \frac{1}{1-x}.$$

Solve this equation for $P_{132}(x,y)$ to obtain the desired result.

The derivation of $P_{123}(x,y)$ is similar, using

$$P_{123}(x,y) = x^2 y P_{123}(x,y) + x^4 y^2 \left(\binom{3}{2} \left(\frac{1+x}{1 - x^2 y}\right)^2 - \binom{3}{1} \left(\frac{1+x}{1 - x^2 y}\right) + 1 \right).$$

□

Combining Proposition 5.10 with Theorem 5.6 allows one to compute $P_{[k]\ominus\pi}(x,y)$ for various π of length 2 or 3. As an example, we next compute $P_{[k]\ominus 12}(x,y)$.

Proposition 5.11 *For all $k \geq 1$ we have*

$$P_{[k]\ominus 12}(x,y) = \frac{\sum_{j=1}^{k-1} w_{j-1}(x,y)\left((1-x^2y)V_{j-1}(x,y) - x\sqrt{y}(1+x)V_{j-2}(x,y)\right)^2}{(1-x^2y)\left((1-x^2y)V_k(x,y) - x\sqrt{y}(1+x)V_{k-1}(x,y)\right)^2}. \qquad (30)$$

Here $w_{2k}(x,y) = 1 + x - x^{2k+1}y^k - x^{2k+2}y^{k+1}$ and $w_{2k-1}(x,y) = 1 + x - x^{2k}y^k - x^{2k+1}y^k$.

Proof. To prove (30) we argue by induction on k. The case $k = 1$ is immediate from Proposition 5.10, so suppose $k \geq 2$ and the result holds for $k-1$. Use Theorems 5.4 and 5.6 to find that

$$\begin{aligned} &P_{[k]\ominus 12}(x,y) = \\ &\quad xP_{[k]\ominus 12}(x,y) + x^2yP_{[k-1]\ominus 12}(x,y)F_{[k]\ominus 12}(x,y) + x^2yF_{[k-1]\ominus 12}(x,y)P_{[k]\ominus 12}(x,y) \\ &\quad +x^4y^2\frac{w_{k-2}(x,y)}{1-x^2y}\left(F_{[k-1]\ominus 12}(x,y)\right)^2 F_{[k]\ominus 12}(x,y). \end{aligned}$$

Solve this equation for $P_{[k]\ominus 12}(x,y)$, use (25) to eliminate $F_{[k]\ominus 12}(x,y)$ and $F_{[k-1]\ominus 12}(x,y)$, and use induction to eliminate $P_{[k-1]\ominus 12}(x,y)$, obtaining (30). □

We conclude this section by combining several of our results concerning $P_\pi(x,y)$ to compute $P_{[k]\ominus 231}(x,y)$.

Proposition 5.12 *For all $k \geq 2$ we have*

$$P_{[k]\ominus 231}(x,y) = \frac{(1+x)\left(W_{k-2}\right)^2 + \sum_{j=1}^{k-2} w_{j-1}(x,y)\left(W_{j-1}\right)^2}{(1-x^2y)\left(W_k\right)^2}.$$

Here

$$\begin{aligned} W_j(x,y) &= (1-x^2y)V_j - x\sqrt{y}(1+x)V_{j-1}, \\ w_{2k}(x,y) &= 1 + x - x^{2k+1}y^k - x^{2k+2}y^{k+1}, \end{aligned}$$

and

$$w_{2k-1}(x,y) = 1 + x - x^{2k}y^k - x^{2k+1}y^k.$$

Proof. It is routine using (10) and the analogue of (9) in which $1 \ominus \pi \ominus 1$ is replaced with $1 \ominus \pi$ to show that

$$F_{[k]\ominus 231}(x,y) = \frac{(1-x^2y)V_{k-1} - x\sqrt{y}(1+x)V_{k-2}}{x\sqrt{y}\left((1-x^2y)V_k - x\sqrt{y}(1+x)V_{k-1}\right)}. \qquad (31)$$

Now set $\pi = [k-1] \ominus 12$ in (29) and use (25), (30), and (31) to simplify the result. □

In view of Conjecture 3.14 one might conjecture that $P_{[l_1,\ldots,l_m]}(x,y)$ is symmetric in $l_1,\ldots,l_m$. However, it is not difficult using Maple to verify that the coefficient of x^7 in $P_{[2,1,1]}(x,y)$ is $14y^2 + 24y^3$ and the coefficient of x^7 in $P_{[1,2,1]}(x,y)$ is $11y^2 + 26y^3$, from which it follows that $P_{[2,1,1]}(x,y) \neq P_{[1,2,1]}(x,y)$, and even that $P_{[2,1,1]}(x,1) \neq P_{[1,2,1]}(x,1)$.

6 Involutions Which Contain 3412 and Another Pattern Exactly Once

For any permutation π and any $n \geq 0$, let $I_n(;3412,\pi)$ (resp. $I(;3412,\pi)$) denote the set of involutions of length n (resp. of any length) which contain exactly one subsequence of type 3412 and exactly one subsequence of type π. In this section we begin a study of the generating function

$$Q_\pi(x,y) = \sum_{\sigma \in I(;3412,\pi)} x^{|\sigma|} y^{tc(\sigma)}.$$

It is possible to obtain results similar to the results of previous sections, but the expressions involved become cumbersome quickly. For this reason we restrict our attention to $Q_{[k]}(x,y)$.

Theorem 6.1 *For all $k \geq 2$ we have*

$$Q_{[2k]}(x,y) = \frac{2x\sqrt{y}}{(1-x^2y)V_k^2} \sum_{i=2}^{k} \frac{w_{2i-3}V_{i-2}V_i + \sum_{j=0}^{i-2} w_{2j+1}V_j^2}{V_{i-1}V_i}$$

and

$$Q_{[2k+1]}(x,y) =$$

$$\frac{2\sqrt{y}}{(1-x^2y)\left(\sqrt{y}V_{k+1}+V_k\right)^2} \sum_{i=1}^{k} \frac{(\sqrt{y}V_{i-1}+V_{i-2})(\sqrt{y}V_{i+1}+V_i)w_{2i-2} + \sum_{j=0}^{i-1}(\sqrt{y}V_j+V_{j-1})^2 w_{2j}}{(\sqrt{y}V_i+V_{i-1})(\sqrt{y}V_{i+1}+V_i)}.$$

Here $w_{2k}(x,y) = 1+x-x^{2k+1}y^k - x^{2k+2}y^{k+1}$ and $w_{2k-1}(x,y) = 1+x-x^{2k}y^k - x^{2k+1}y^k$.

Proof. In view of Theorems 5.4 and 5.6 we have

$$\begin{aligned}
Q_{[k]}(x,y) &= \\
& xQ_{[k]}(x,y) + x^2yQ_{[k-2]}(x,y)F_{[k]}(x,y) + x^2yP_{[k-2]}(x,y)G_{[k]}(x,y) \\
& + x^2yF_{[k-2]}(x,y)Q_{[k]}(x,y) + x^2yG_{[k-2]}(x,y)P_{[k]}(x,y) \\
& + x^4y^2\left(\frac{w_{k-3}(x,y)}{1-x^2y}\right)\left(2G_{[k-2]}(x,y)F_{[k-2]}(x,y)F_{[k]}(x,y) + G_{[k]}(x,y)\left(F_{[k-2]}(x,y)\right)^2\right).
\end{aligned}$$

Solve this equation for $Q_{[k]}(x,y)$ and use (9) to obtain a recurrence relation for $Q_{[k]}(x,y)$. Now the result follows by induction on k, using Theorem 3.8, Proposition 5.8, the fact that $Q_{[1]}(x,y) = Q_{[2]}(x,y) = 0$, the fact that

$$G_{[2k]}(x,y) = \frac{1}{V_k^2(x,y)},$$

and the fact that

$$G_{[2k-1]}(x,y) = \frac{1}{x\left(\sqrt{y}V_k(x,y)+V_{k-1}(x,y)\right)^2}.$$

□

Theorem 6.1 enables us to find $|I_n(;3412,[k])|$ for various k. For instance, for all $n \geq 6$ we have

$$|I_n(;3412,4321)| = (3n^2 - 23n + 38)2^{n-8}.$$

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