An Introduction to Symmetric Functions and Their Combinatorics

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Commentary

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Below is a collection of comments and clarifications for my book *An Introduction to Symmetric Functions and Their Combinatorics*. This list is very short right now, because it does not yet reflect all of the comments and clarifications I have received. However, I am continually adding to it, and I will post an updated version every few weeks. In addition, I may have misunderstood (or mistyped) something on this list, so please let me know if anything here looks incorrect.

I would like to thank everyone who has sent me comments and clarifications. I would especially like to thank Darij Grinberg for his helpful feedback, and for sharing his extensive list of corrections and comments.

The Definition of a Symmetric Function

In our definition of a symmetric function (Definition 1.19, on page 17) we allow formal power series of unbounded degree. This is nonstandard: most authors only allow a power series which is invariant under all permutations of the underlying set of variables to be called a symmetric function if the degrees of its terms are bounded. Thus, for example, $\prod_{j=1}^{\infty}(1 + x_j)$ is a symmetric function under our definition but not under the standard definition. For most of the material in the book this distinction won't make a difference, and there are examples in Chapter 3 of interesting objects which are symmetric functions under our definition but not under the standard definition. We do have to take care when we study the Hall inner product: if we allow unbounded degree then inner products of some pairs of symmetric functions result in divergent series, and are therefore undefined.

Infinite Products

In Problem 1.21 (and in the comment above) we have two infinite products of polynomials, so one might wonder how we can make sense of a product like this, and in particular, how we define convergence for these products. For us, an infinite product $\prod_{j=1}^{\infty} f_j(X)$, in which each factor is a polynomial in $x_1, x_2, \ldots$, converges when we can determine the coefficient of each monomial in $x_1, x_2, \ldots$. More precisely, if z is a monomial in $x_1, x_2, \ldots$, then we write $[z]f(X)$ to denote the coefficient of z in the formal power series $f(X)$. We say $\prod_{j=1}^{\infty} f_j(X)$ converges whenever the sequence $[z]f_1(X), [z](f_1(X)f_2(X)), [z](f_1(X)f_2(X)f_3(X)), \ldots$ is eventually constant for every monomial z. This constant is the coefficient of z in $\prod_{j=1}^{\infty} f_j(X)$. Note that this is the definition used by Stanley [Sta11, p. 7].

In the products in Problem 1.21, each variable appears in finitely many factors, so the coefficient of each monomial is constant once every factor in which any of its variables appears is included in the partial product. Therefore, the products both converge.

Problem 2.9 and the Gale-Ryser Theorem

The result in Problem 2.9 is the Gale-Ryser theorem, and proving it may be more difficult than solving most of the other problems in this chapter. This result is sometimes stated in terms of the existence of a matrix of 0s and 1s with specified row and column sums, and sometimes in terms of the existence of a bipartite graph with specified degree sequences. Proofs are available in several places, including [Bru06, p. 27] and [Kra96].

A Non-Combinatorial Proof of Proposition 3.3

We can use (3.5) and (2.12) to give an algebraic proof of (3.6), as follows.

$$\begin{aligned}
\sum_{j=1}^{k} h_{k-j}p_j &= \sum_{j=1}^{k} h_{k-j} \sum_{m=1}^{j} (-1)^{j+1} m e_{j-m} h_m && \text{(by (3.5))} \\
&= \sum_{m=1}^{k} m h_m \sum_{j=m}^{k} (-1)^{j+m} h_{k-j} e_{j-m} \\
&= \sum_{m=1}^{k} m h_m \sum_{l=0}^{k-m} (-1)^{l} h_{k-m-l} e_{l} \\
&= k h_k && \text{(by (2.12))}
\end{aligned}$$

We can give a similar proof of (3.7) using (3.4) and (2.12).

References

[Bru06] Richard A. Brualdi. *Combinatorial Matrix Classes*. Cambridge University Press, 2006.

[Kra96] Manfred Krause. A simple proof of the Gale-Ryser theorem. *Amer. Math. Monthly*, 103(4):335–337, 1996.

[Sta11] Richard P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, second edition, 2011.